



Principles of Finance

Summer Term 2005

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Course Grade

- You have to prepare the assigned **exercises**. In total you have to prepare and mark at least 50% of all exercises. A sheet will be posted in front of my door and I will then randomly call students to present their work. Just writing down results is not enough! If you can not explain what you have done, you will lose 10% of the whole course. If you are caught again, you will lose another 20%.
- **Exercises**: A maximum of 5 points per exercise presented can be reached.
- **Homework** due 18.01.2005: 10 points.
- **Midterm**, 30.11.2005, 12-14, HS 2: 40 points.
- **Final**, 25.01.2006, 12-14, HS 2: 50 points.
- To pass the course you need at least 50 points in total and you have to mark at least 50% of all exercises!



Syllabus

Part 1 - Single-period random cash flows

Stocks (incl. empirical features of returns)

Mean-variance portfolio theory

Utility theory

“Capital Asset Pricing Model” (incl. performance measurement)

Factor models (incl. “Arbitrage Pricing Theory”)

Part 2 - Multi-period deterministic cash flows

Fixed income securities (incl. credit and market risk)

Floating rate notes

Midterm

Part 3 - Derivative securities

Forwards

Futures

Options

Swaps



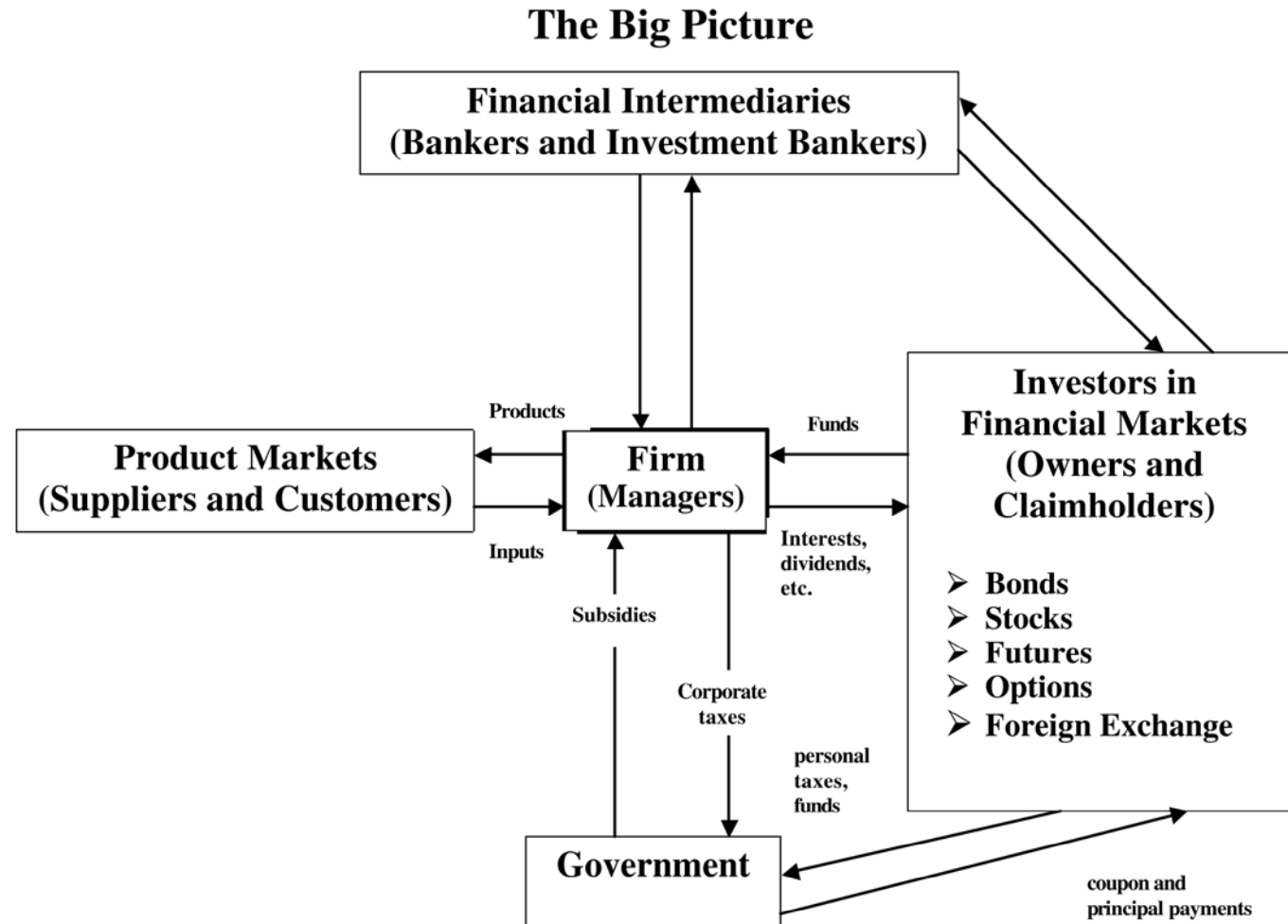
Literature

- First Part:
Investment Science, David Luenberger, Oxford University Press
- Second Part:
Options, Futures, and Other Derivatives, John Hull, 6th edition, Prentice Hall

Overview

- Financial markets and the firm
- Players in financial markets
- Products traded on financial markets
- Classification of financial markets
- Pricing

Financial markets and the firm



Source: Admati (2002)

Players in financial markets

- **Borrowers:** need funds
- **Lenders / investors:** wish to invest funds
- **Hedgers:** want to reduce risk
- **Speculators:** are willing to take risk
- **Arbitrageurs:** lock in profits by exploiting market inefficiencies
 - **Arbitrage opportunity / profit:** riskless profit with zero initial investment
 - **Arbitrage strategy:** buy cheap and sell expensive
- **Financial Intermediaries (FI)**



Players in financial markets: FI

- ... matching borrowers and lenders / investors
- Ex-post **information asymmetry** between potential lenders and a risk neutral entrepreneur and **costly monitoring** ↻ FI (**commercial banks**) are optimal (least costly alternative) given a “high” number of lenders (see Diamond (1984))
- Other FI
 - **Investment banks:** help companies to obtain funding directly from lenders
 - **Brokers:** match investors wishing to trade with each other
 - **Market makers:** have the commitment to buy and sell from or to investors



Products traded on financial markets

- **Bonds / FI securities (deterministic CF stream):** e.g. classification according to issuer: government bonds and corporate bonds
- **Shares (random CFs):** common stock and preferred stock
- **Derivatives:** forwards, futures, swaps and options
- **Currencies / foreign exchange (FX)**
- **Commodities**

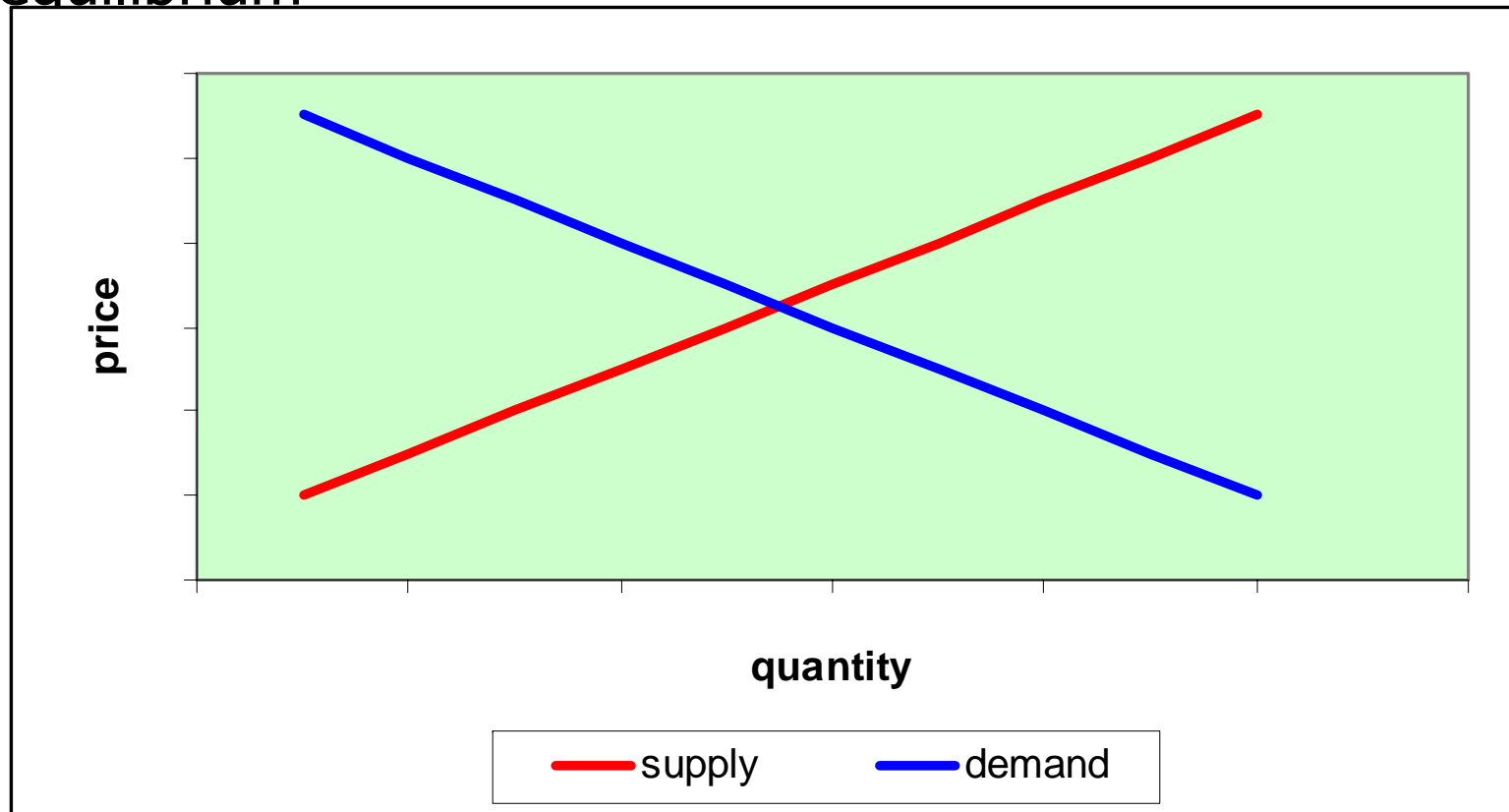


Classification of financial markets

- ... according to **types of markets / traded products**: bond market, stock market, derivatives market, FX market, commodities market
- ... according to **investor's horizon**: money market (spot market) vs. capital market (future market)
- **Issuance vs. trading of securities**: primary market vs. secondary market
- ... according to the **trading system**: auction market, dealer market and hybrid systems (... combination of auction and dealer market)

Pricing

- **Supply and demand** ↻ price and quantity in an equilibrium

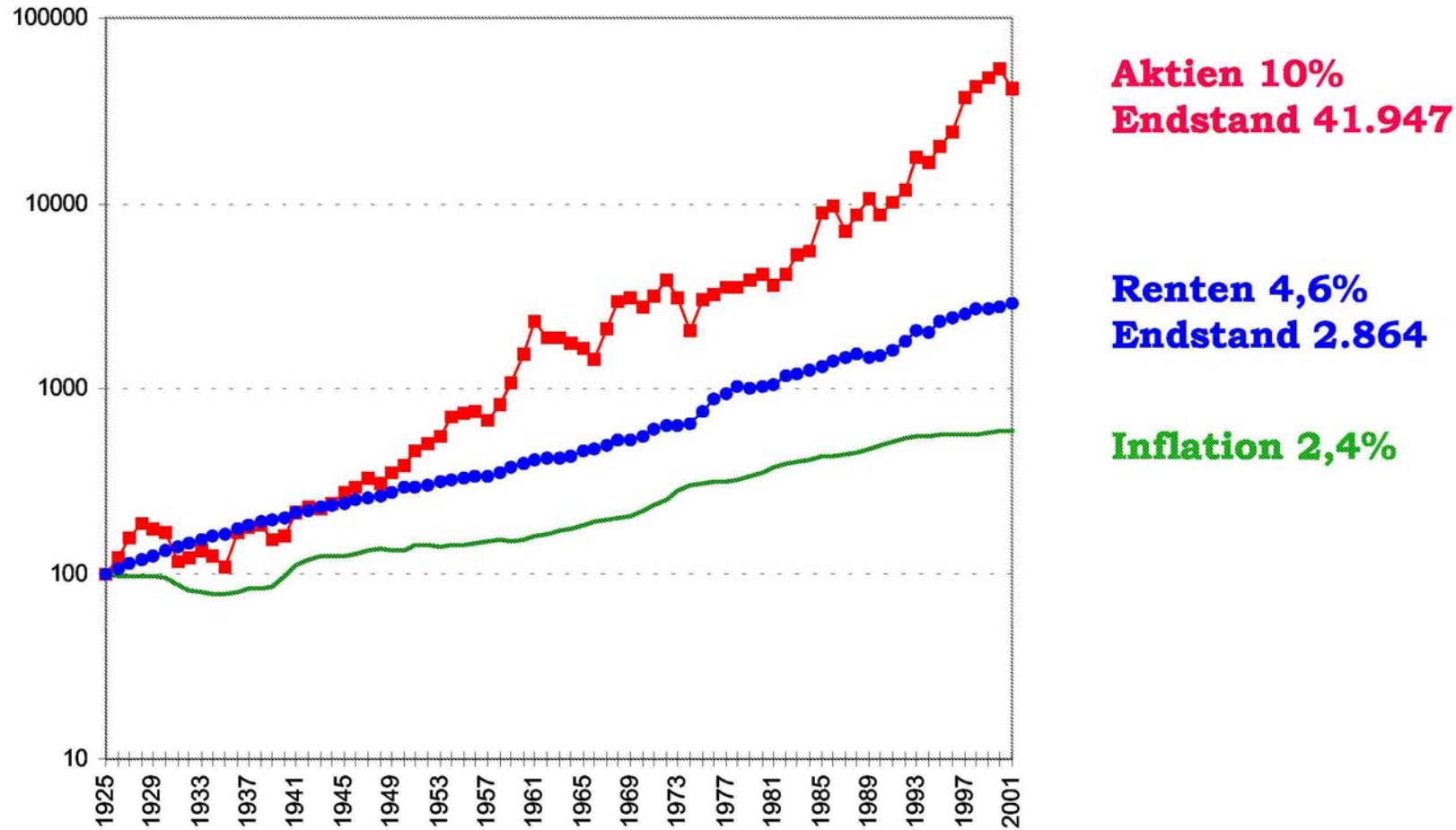


Pricing

- What determines the **price** at which investors are willing to trade?
 - Expectations about future cash flows
 - Timing of these cash flows
 - Riskiness of these cash flows
 - ↻ Present value of future CFs
- **Valuation axioms**
 - Investors prefer more to less
 - Investors are risk averse
 - Money has a time value
 - Investors are rational

Pricing

- **Returns** for different asset classes in Switzerland




Source: Spremann (2002)

Overview

- Important stock markets
- Types of stocks
- Types of trades
- Computing returns
- Empirical features of returns
- Additional literature
 - Haugen, R. A., *Modern Investment Theory*, 5th edition, 2001. Relevant chapter: 2.
 - Grinblatt, M. and Titman, S., *Financial Markets and Corporate Strategy*, 2nd edition, 2002. Relevant chapter: 4.
 - Bleymüller, J., Gehlert, G., and Gülicher, H., *Statistik im Studium der Wirtschaftswissenschaften*, 10th edition, 1996. Relevant chapters: 3 and 15.

Important stock markets

- Market capitalisation in million USD.

Source: 

Exchange	End 1990	End 1995	End 2000
NYSE	2 692 123	5 654 815	11 534 613
Nasdaq	310 800	1 159 940	3 597 086
Japan (Tokyo)	2 928 534	3 545 307	3 157 222
London	850 012	1 346 641	2 612 230
Euronext Paris	311 687	499 990	1 446 634
Deutsche Börse	355 311	577 365	1 270 243
Switzerland	157 635	398 088	792 316
Toronto	241 924	366 345	770 116
Italy	148 766	209 522	768 363
Euronext Amsterdam	119 825	286 651	640 456
Helsinki	22 721	44 137	293 635
Vienna	26 320	32 513	29 935
Ljubljana	-	297	3 100

Types of stocks

▪ **Common stock**

- ... residual claim on the earnings of the firm
- Common stockholders can expect to receive their income as
 - (common) dividends and/or
 - capital gains
- In general, firms that pay out only a small fraction of earnings as common dividends can be expected to grow faster than firms that pay out a larger fraction

▪ **Preferred stocks**

- ... mixture between fixed and variable income security
- Preferred stocks are usually perpetual securities having no maturity date, although there are exceptions (however, preferred stocks are usually callable)
- **Cumulative** versus **noncumulative** preferred stock



Types of trades

- Classification on the basis of the **execution price**
 - **Market order:** executed at the best available price
 - **Limit order:** executed at a price at least as advantageous as a stated price (if the trade can't be completed at that price, it is delayed until it is possible to execute it under those conditions)
 - **Stop loss order:** sell if the price falls to a specified level
- Classification on the basis of **allowable time for completion**
 - **Good until canceled:** remains indefinitely
 - **Good until date:** remains valid until a prespecified date
 - **Good for day / day order:** must be executed by the end of the day or it is canceled
 - **Fill or kill order:** must be executed immediately or it is canceled



Types of trades

- **Long position:** owning an asset (e.g. 100 OMV shares)
- **Short position / short selling**
 - Borrow shares from someone (the owner) usually through a broker, i.e. taking a short position
 - Sell (short) these shares, say for x
 - Pay dividends to the owner of the shares
 - Buy shares back, say for y
 - Return the shares borrowed, i.e. closing out the short position
 - Profit / loss = $x - y - \text{dividends paid}$
- If the owner wants to sell her shares the broker will simply borrow them from some other customer. However, if there are too many short sales and not enough customers from whom to borrow shares, the broker may fail to execute the trade ("**short squeeze**"). In a short squeeze the broker has the right to force us to close out our short position.

Computing returns

- **Simple returns, discrete compounding**

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

- **Log returns, continuous compounding**

$$y_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln P_t - \ln P_{t-1}$$

- Relation between simple and log returns

$$r_t = e^{y_t} - 1 \Leftrightarrow y_t = \ln(1 + r_t)$$

Computing returns

- Multi-period simple returns

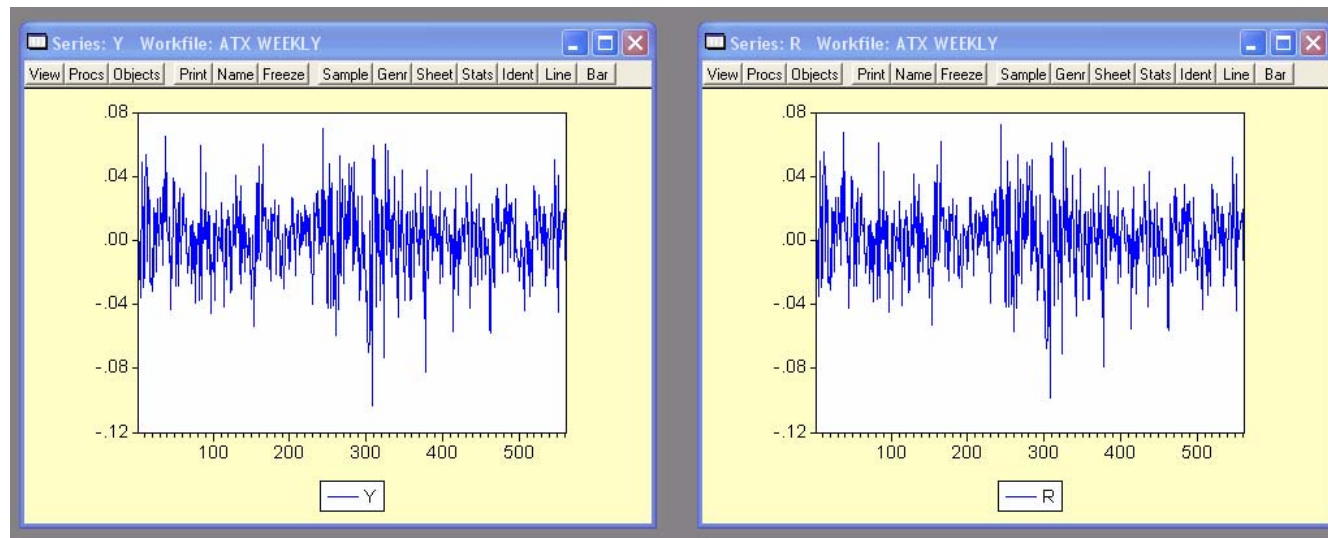
$$\begin{aligned} 1 + r_t(h) &= \frac{P_t}{P_{t-h}} = \\ &= \frac{P_t}{P_{t-1}} \frac{P_{t-1}}{P_{t-2}} \dots \frac{P_{t-h+1}}{P_{t-h}} = \\ &= (1 + r_t)(1 + r_{t-1}) \dots (1 + r_{t-h+1}) = \prod_{j=0}^{h-1} (1 + r_{t-j}) \end{aligned}$$

- Multi-period log returns

$$\begin{aligned} y_t(h) &= \ln P_t - \ln P_{t-h} = \\ &= (\ln P_t - \ln P_{t-1}) + (\ln P_{t-1} - \ln P_{t-2}) + \dots + (\ln P_{t-h+1} - \ln P_{t-h}) = \\ &= y_t + y_{t-1} + \dots + y_{t-h+1} = \sum_{j=0}^{h-1} y_{t-j} \end{aligned}$$

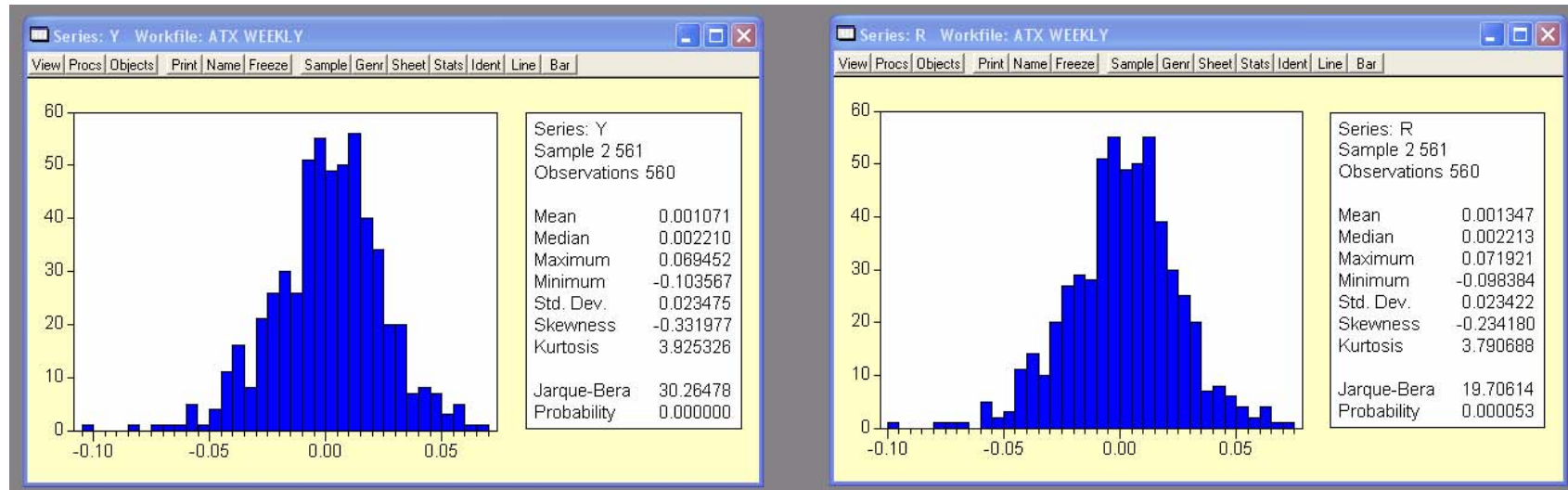


Empirical features of returns



- Simple and log returns cannot be distinguished in such graphs
- Erratic (“**white noise**”), strongly oscillating behavior of returns around the more or less constant mean (“**stationary process**” i.p. “**mean reverting**”)
- Variance / volatility (standard deviation) is not constant over time (“**heteroskedasticity**”) i.p. we have periods of different length with approximately the same degree of variation (“**volatility clustering**”)

Empirical features of returns



- Usually returns **aren't normal distributed!**
 - Skewness $\neq 0$
 - Kurtosis $\neq 3$, usually kurtosis > 3 ("leptokurtic", i.e. the distribution is more strongly concentrated around the mean than the normal and assigns correspondingly higher probabilities to extreme values; **fat tails**)
- ☹ Prices aren't lognormal distributed!

Overview

- Properties of portfolios
 - Return
 - Risk
 - Diversification

- Minimum variance and efficient set
 - Combination lines
 - Markowitz
 - Tobin

- Additional literature
 - Haugen, R. A., *Modern Investment Theory*, 5th edition, 2001.
Relevant chapters: 4 and 5.

Properties of portfolios: return

▪ Multiple-asset portfolio

$$r_P = \sum_{j=1}^n x_j r_j$$

x_j = Dollar amount in of security J bought (or sold short) / total equity investment in the portfolio

- ... portfolio return is weighted average of individual securities' returns
 - Weights are fractions of individual securities in total portfolio value
 - Weights can be positive (long position) or negative (short position)
 - Weights must add to 1

▪ Expected value of a random variable

$$25 \quad E[X] = \sum_s x_s p_s = \int_{-\infty}^{\infty} x f(x) dx \Rightarrow E[r_P] = \sum_{j=1}^n x_j E[r_j]$$

Properties of portfolios: risk

- We quantify risk in terms of statistical measures, conventionally this is done using the **variance / standard deviation (volatility)**

- Variance of a random variable

$$\sigma_Y^2 = V[Y] = E[(Y - E[Y])^2] = E[Y^2] - E[Y]^2$$

- Covariance and correlation of two random variables

$$\sigma_{yx} = \text{cov}[YX] = E[(Y - E[Y])(X - E[X])] = E[YX] - E[Y]E[X]$$

$$\rho_{Y,X} = \frac{\sigma_{Y,X}}{\sigma_Y \sigma_X}, \quad -1 \leq \rho_{Y,X} \leq 1$$

- Variance of a weighted sum

$$\sigma_{aY+bX}^2 = V[aY + bX] = a^2V[Y] + b^2V[X] + 2ab \text{cov}[YX]$$

Properties of portfolios: risk

▪ Multiple-asset portfolio

$$\sigma_P^2 = \sum_{i=1}^n \sum_{j=1}^n x_i \cdot x_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{i,j} = x' \Sigma x$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$$

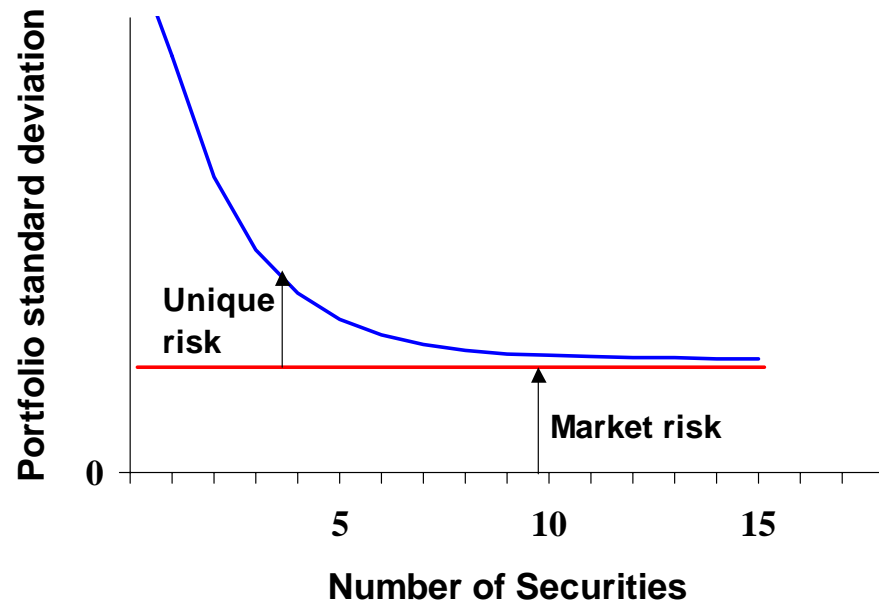
Examples

- An investor has € 1000. Hearing from an investment opportunity with an expected rate of return of 24%, she sells short another security with an expected return of 5% for € 4000 and invests all his money in the other security. What is the expected rate of return on the portfolio?
- Given are two uncorrelated securities: Stock A with $E(r)=12\%$, $SD(r)=8\%$ and stock B with $E(r)=2\%$, $SD(r)=10\%$. Calculate the expected rate of return and standard deviation for a portfolio of € 15000 long in A and € 5000 short in B.



Properties of portfolios: diversification

- **Diversification:** strategy designed to reduce risk by spreading the portfolio across many assets



- **Unique risk / unsystematic risk / diversifiable risk / idiosyncratic risk:** risk factors affecting only that firm
- **Market risk / systematic risk:** economy-wide sources of risk that affect the overall stock market

Properties of portfolios: diversification

- **Naive diversification:** portfolio with n assets, each asset with weight $1/n$

Example (2 years of recent weekly data): naive portfolios of Austrian stocks

Boehler	Lenzing	Mayr MK	Erste	EVN	Return	#	SD
100.00					0.033	1	0.219
50.00	50.00				0.106	2	0.145
33.33	33.33	33.33			0.117	3	0.145
25.00	25.00	25.00	25.00		0.153	4	0.135
20.00	20.00	20.00	20.00	20.00	0.140	5	0.121



Minimum variance and efficient set

- A better method for diversification
 - Find out the portfolio weights that minimize the portfolio variance for a given expected portfolio return
 - For any two assets, plotting return and standard deviation for all feasible portfolio weights yields the **combination line** for these assets

- Assume the following expected returns and standard deviations for two **uncorrelated** securities:

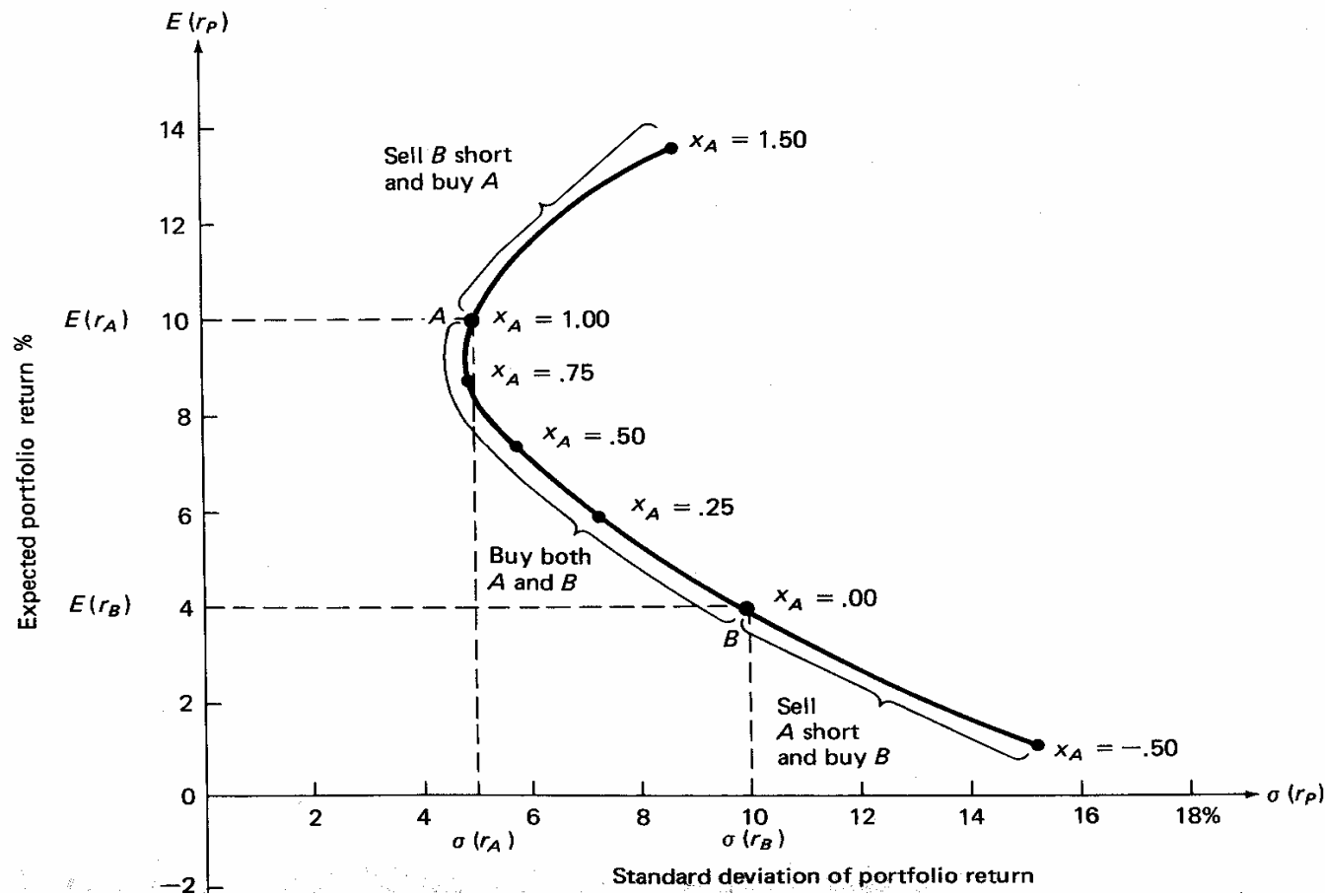
	Security	
	A	B
E(r)	0.10	0.04
SD(r)	0.05	0.10

$$E[r_P] = x_A 0,10 + (1 - x_A) 0,04$$

$$\sigma(r_P) = \sqrt{x_A^2 0,05^2 + (1 - x_A)^2 0,10^2}$$

Minimum variance and efficient set

↻ Combination line



Minimum variance and efficient set

- The case of **perfect positive correlation**

	Security	
	A	B
E(r)	0.10	0.04
SD(r)	0.05	0.10

$$E(r_P) = x_A 0,10 + (1 - x_A) 0,04$$

$$\sigma^2(r_P) = x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2x_A(1 - x_A)\sigma_A\sigma_B$$

$$\sigma^2(r_P) = (x_A \sigma_A + (1 - x_A) \sigma_B)^2$$

$$\sigma(r_P) = |x_A 0,05 + (1 - x_A) 0,10|$$

Minimum variance and efficient set

- The case of **perfect negative correlation**

	Security	
	A	B
E(r)	0.10	0.04
SD(r)	0.05	0.10

$$E(r_P) = x_A 0,10 + (1 - x_A) 0,04$$

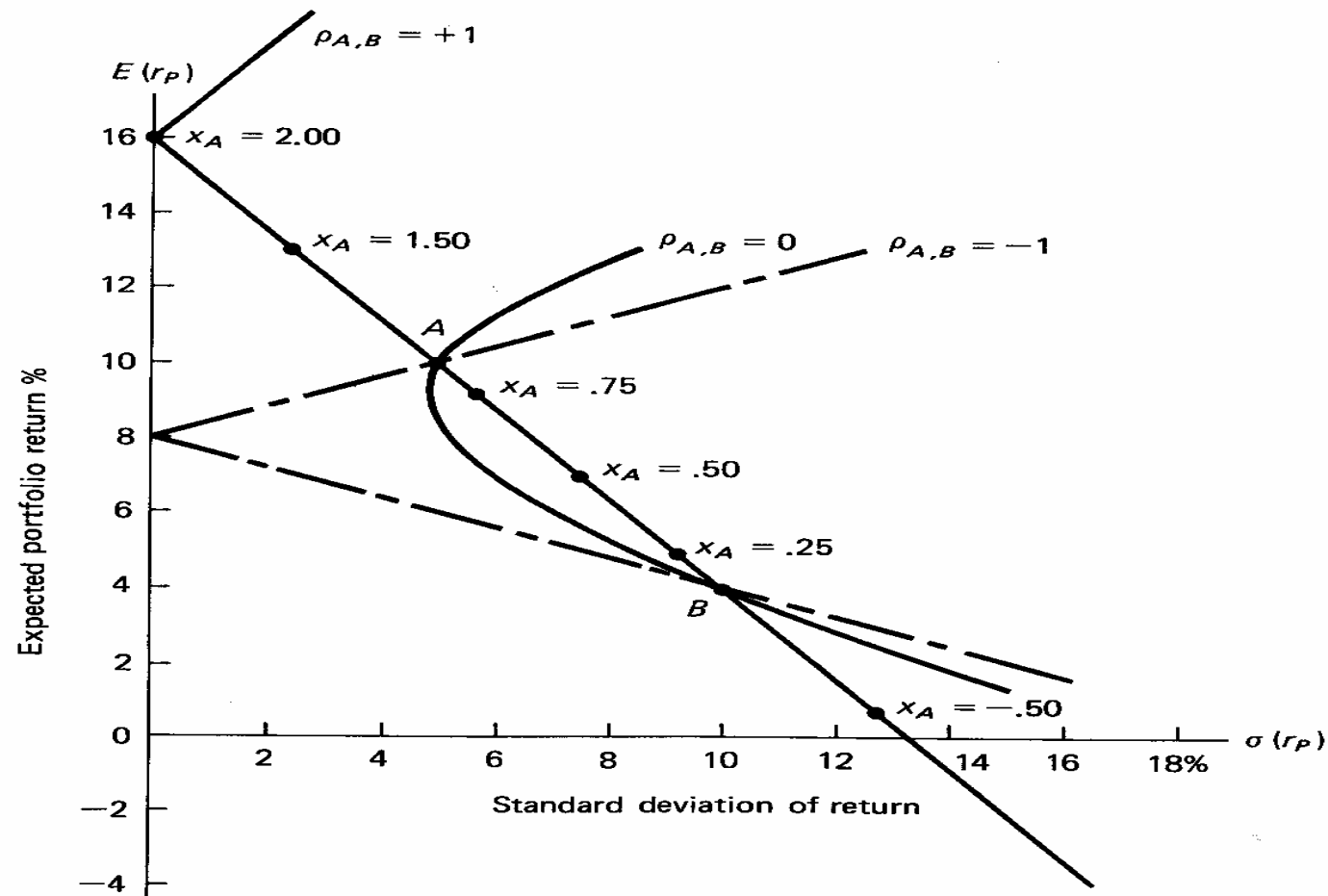
$$\sigma^2(r_P) = x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 - 2x_A(1 - x_A)\sigma_A\sigma_B$$

$$\sigma^2(r_P) = (x_A \sigma_A - (1 - x_A)\sigma_B)^2$$

$$\sigma(r_P) = |x_A 0,05 - (1 - x_A) 0,10|$$

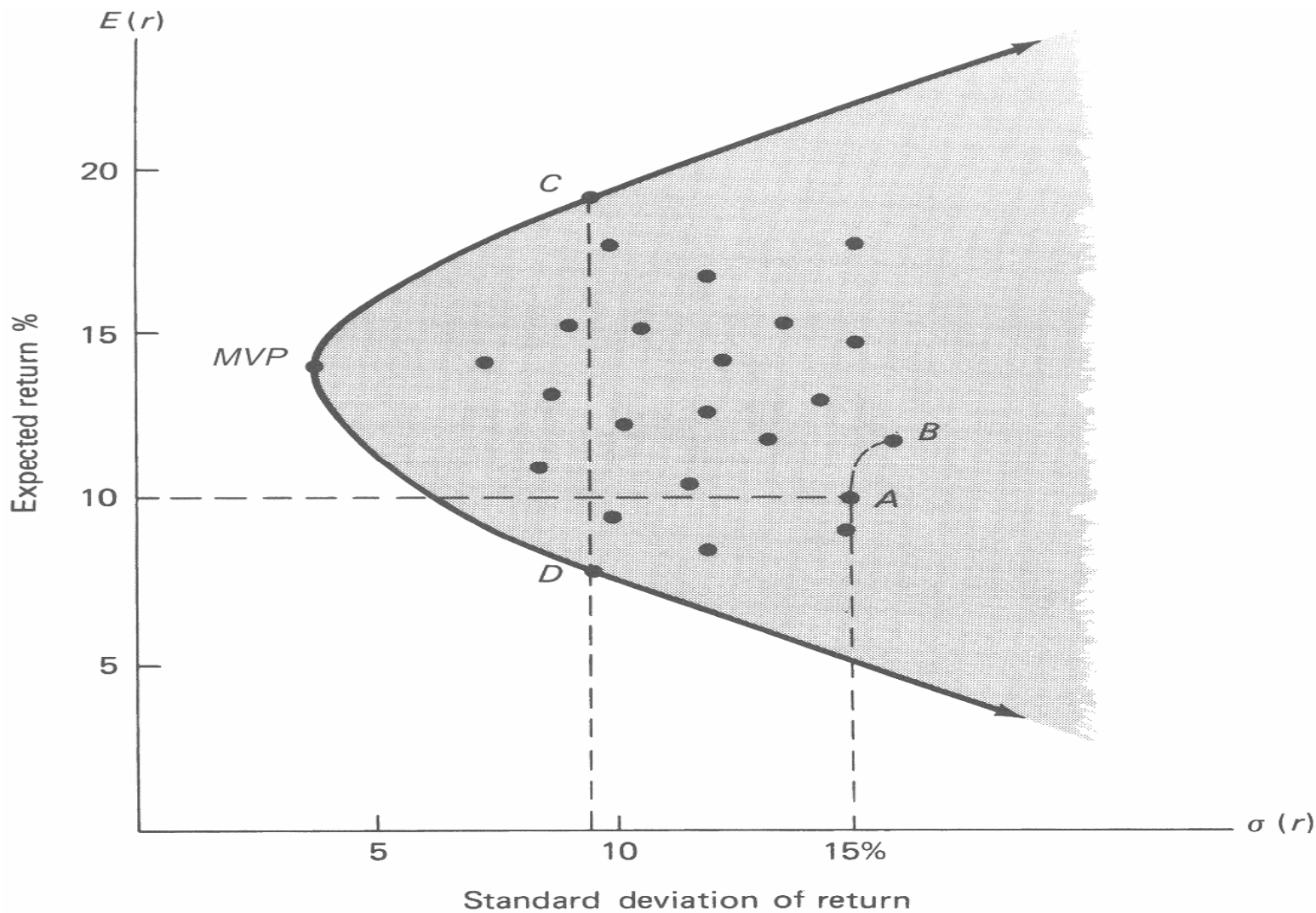
Minimum variance and efficient set

- ◉ Spectrum of combination lines





Minimum variance and efficient set





Minimum variance and efficient set

- Given a particular level of expected rate of return, the portfolio on the **minimum variance set** (“bullet”) has the lowest standard deviation (or variance) achievable with the available population of stocks
- The portfolio with the lowest possible level of standard deviation is called global **minimum variance portfolio (MVP)**, it divides the bullet in two halves
- The top half of the “bullet” is called the **efficient set / - frontier** (the portfolios in the efficient set have the highest attainable expected rate of return for a given level of standard deviation)



Minimum variance and efficient set

- Finding the efficient set using Lagrange - **Markowitz model**: compute the portfolio weights that minimize the portfolio variance for a given expected portfolio return

$$\min_x \sigma_P^2 = \sum_{i=1}^n \sum_{j=1}^n x_i \cdot x_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{i,j} = x' \Sigma x$$

s.t.

$$(i) \ r_p^* = \sum_{j=1}^n x_j \bar{r}_j$$

$$(ii) \ 1 = \sum_{j=1}^n x_j$$



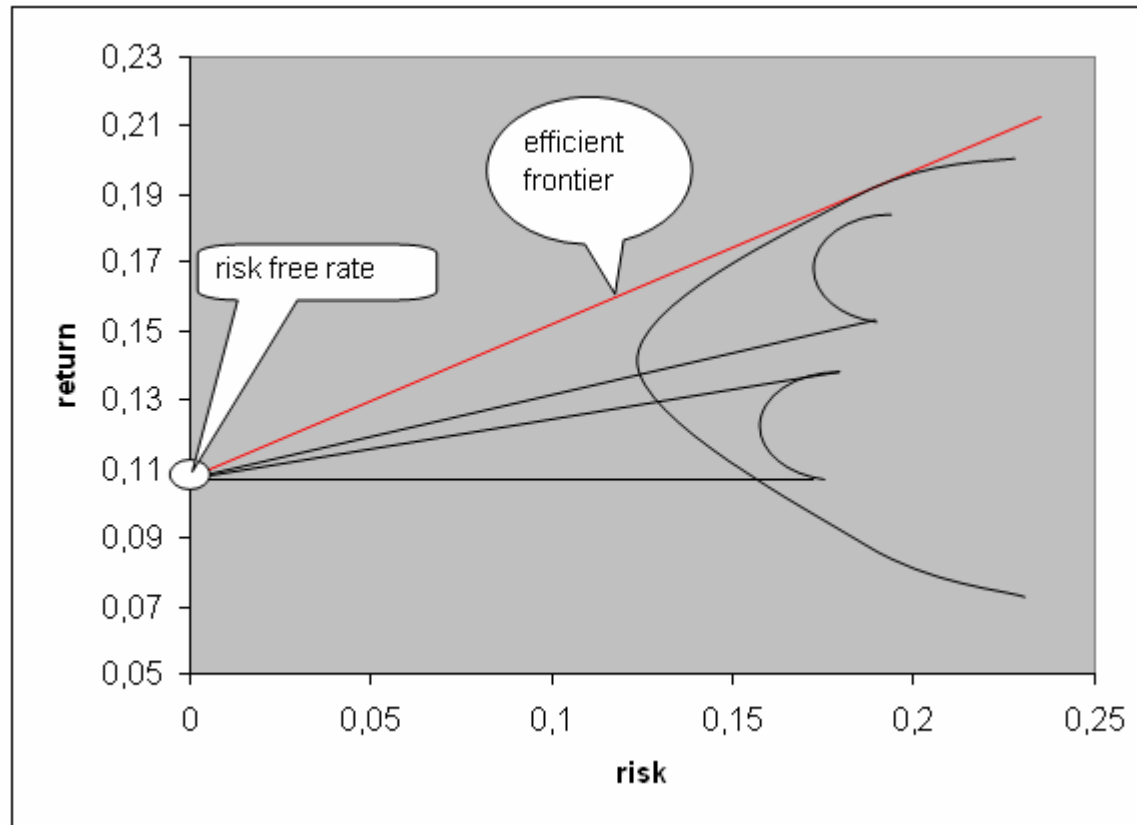
Minimum variance and efficient set

- **Two Fund Separation Theorem:** combinations of portfolios on the minimum variance set are again on the minimum variance set
- Remarks
 - Once you found any two funds on the efficient set, it is possible to create all other mean-variance efficient portfolios from these 2 funds ↻ there is no need for anyone to purchase individual stocks separately!
 - It suffices to replicate mean and variance, since the prices of portfolios with the same mean and the same variance have to be the same (law of one price)!



Minimum variance and efficient set

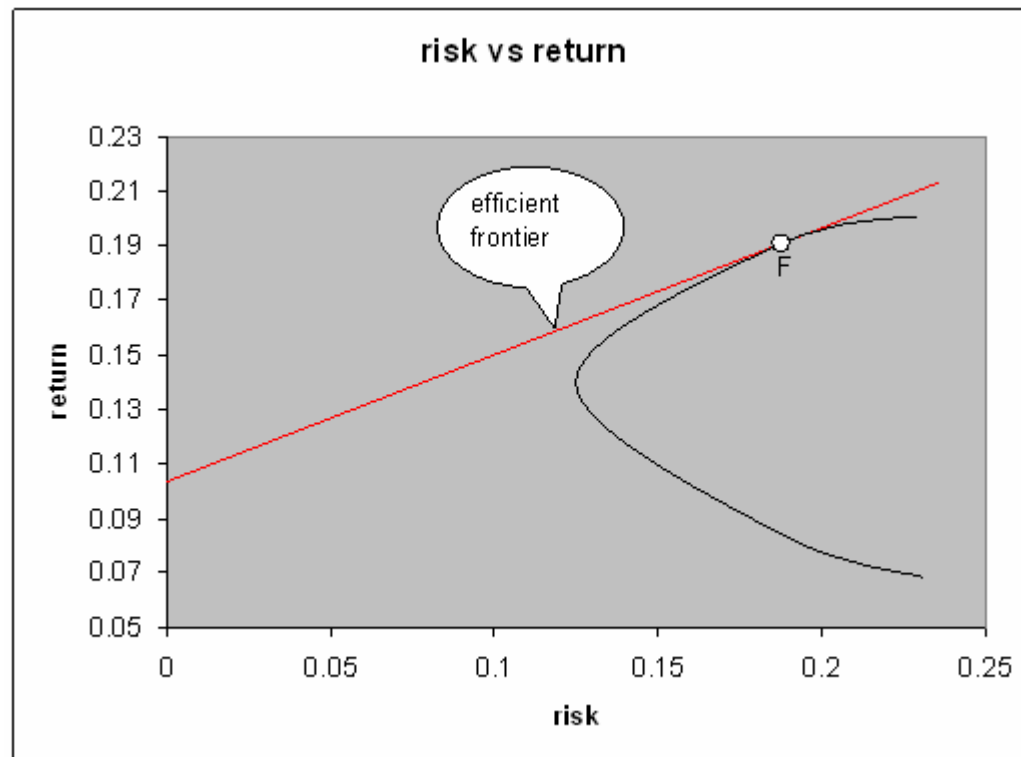
- Including a **risk-free asset**, i.e. include an asset with zero variance and zero covariance to all other assets





Minimum variance and efficient set

- **One Fund Theorem:** there is a single risky portfolio F such that any efficient portfolio can be constructed as a combination of F and the risk-free asset.





Minimum variance and efficient set

- Finding the efficient set using Lagrange - **Tobin model**: compute the portfolio weights that maximize the angle between horizontal axis and the efficient frontier o.e. the tan of this angle

$$\max_x \tan \theta = \frac{\text{excess return}}{\sigma_P} = \frac{\sum_{i=1}^n w_i (\bar{r}_i - r_f)}{\left(\sum_{i=1}^n \sum_{j=1}^n x_i \cdot x_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{i,j} \right)^{0,5}}$$

s.t.

$$1 = \sum_{j=1}^n x_j$$

Overview

- Decisions under uncertainty / Decision criteria
- Utility functions
 - 3 (possible) categories
 - Degree of risk aversion
 - Some utility functions
- Expected value and mean-variance criterion
- Expected utility and mean-variance criterion



Decisions under uncertainty / Decision criteria

▪ Expected value criterion

- **St. Petersburg Paradox** (Daniel **Bernoulli**, 1738): Consider the following game.

A fair coin will be tossed repeatedly until heads comes up. If this happens in the i -th toss, the lottery yields a money prize of 2^i Euros. The probability of this outcome is $1/2^i$.

How much will an expected value maximizer be willing to pay to play this game?

©.

Since one would not suppose, at least intuitively, that real-world people would be willing to pay an infinite amount of money to play this game, the expected value criterion seems to be not appropriate to determine the price to play this game!



Decisions under uncertainty / Decision criteria

- Daniel Bernoulli's solution involved two ideas that have since revolutionized economics:
 - firstly, that people's utility from wealth, $u(w)$, is not linearly related to wealth, w , but rather increases at a decreasing rate - the famous idea of **diminishing marginal utility**, $u'(w) > 0$ and $u''(w) < 0$; and
 - secondly, that a person's valuation of a risky venture is not the expected value, $E[w]$, of that venture, but rather the **expected utility**, $E[u(w)]$, from that venture.

↳ (von Neumann-Morgenstern) expected utility criterion

$$E[u(w)] = \sum_{i=1}^n p_i u(w_i)$$



Decisions under uncertainty / Decision criteria

- With only a handful of exceptions, Bernoulli's expected utility hypothesis was never really picked up until John von Neumann and Oskar Morgenstern's (1944) *Theory of Games and Economic Behavior*.



Utility functions: 3 (possible) categories

- (Strict) risk aversion
 - $u' > 0, (u'' < 0) u'' \leq 0 \dots$ "(strictly) diminishing marginal utility"
 - $(E[u(w)] < u[E(w)]) \quad E[u(w)] \leq u[E(w)]$
... "Jensen's (strict) inequality"
 - Certainty equivalent $(<) \leq E[w]$
 - Risk premium $(>) \geq 0$
 - **(strictly) risk averse iff u is (strictly) concave**
- Risk neutral
 - $u' > 0, u'' = 0 \dots$ "constant marginal utility"
 - $E[u(w)] = u[E(w)]$
 - Certainty equivalent $= E[w]$
 - Risk premium $= 0$
 - **risk neutral iff u is linear**



Utility functions: degree of risk aversion

- u is unique up to a strictly increasing affine transformation (, i.e. u can be replaced by $a+bu$ for any constants a and $b>0$ without changing the preference ordering of u) ↪ such a transformation should not change the measure of risk aversion!

↪ Arrow-Pratt measure of absolute risk aversion

$$A = -\frac{u''}{u'}$$

↪ Risk tolerance

$$T = \frac{1}{A}$$

↪ Arrow-Pratt measure of relative risk aversion

$$R = w^* A = -w \frac{u''}{u'}$$

Some utility functions

- Power utility

$$u(w) = \frac{1}{\gamma - 1} (\alpha + \gamma w)^{1 - \frac{1}{\gamma}}, \text{ for } -\alpha < \gamma w, \gamma \neq 0, \text{ and } \gamma \neq 1, A = \frac{1}{\alpha + \gamma w}$$

- Quadratic utility (special case of power utility: for $\gamma = -1$)

$$u(w) = -\frac{1}{2} (\alpha - w)^2, \text{ for } w < \alpha, A = \frac{1}{\alpha - w}$$

- Negative exponential utility

$$u(w) = -e^{-\alpha w}, \text{ for } \alpha > 0, A = \alpha$$

- Logarithmic utility

$$u(w) = \ln(\alpha + w), \text{ for } -\alpha < w, A = \frac{1}{\alpha + w}$$

- All these utility functions are strictly increasing and strictly concave!*



Expected utility and expected value criterion

- Obviously the expected value criterion can be reconciled with the expected utility approach by using a **linear utility function!**

- $u(w) = w$

$$E[u(w)] = \sum_{i=1}^n p_i u(w_i) = \sum_{i=1}^n p_i w_i$$

- Recall that a linear utility function is equivalent to assuming risk neutrality.
 - Thus, given risk neutrality (i.e. a linear utility function) the expected utility criterion reduces to the expected value criterion.



Expected utility and mean-variance criterion

- The mean-variance criterion used in the Markowitz model can be reconciled with the expected utility approach in either of two ways
 - using a **quadratic utility function**, or
 - assuming **normal returns**.
- Quadratic utility function
 - Utility reaches a maximum at some wealth level and then declines.
 - As your wealth level increases, your willingness to take on risk decreases.
- Normal returns
 - Recall that empirical results reveal that returns aren't normal distributed!

Overview

- Assumptions
- Capital market line (CML)
- Security market line (SML)
- Critique
- Performance measurement
- Additional literature
 - Haugen, R. A., *Modern Investment Theory*, 5th edition, 2001.
Relevant chapters: 8 and 11.

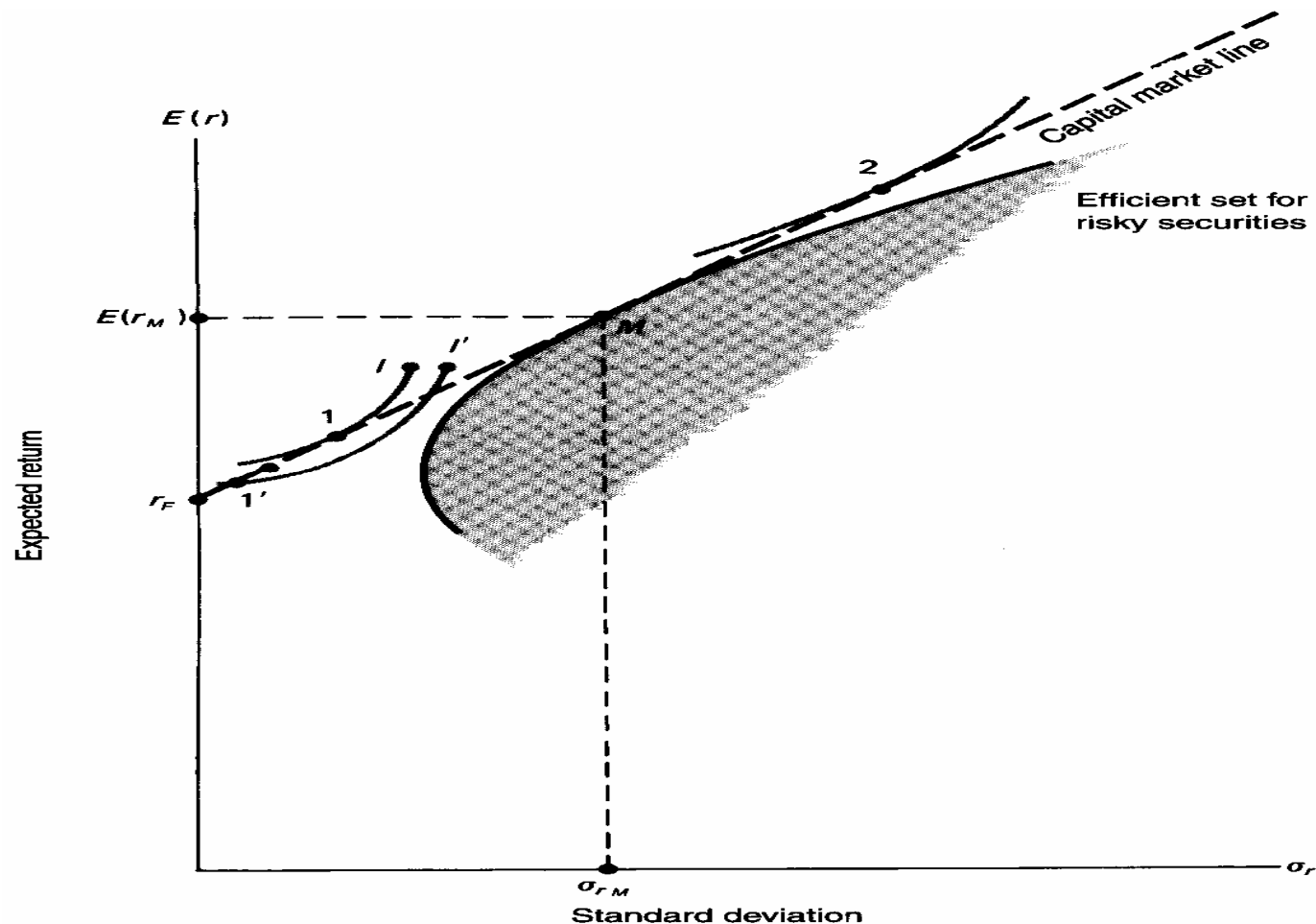


Assumptions

- The CAPM was simultaneously and independently discovered by W. Sharpe (1964), J. Lintner (1965), and J. Mossin (1966).
- **Investors can choose on the basis of expected return and variance!** Recall that this is true if either
 - portfolio returns are normally distributed or
 - investors have a quadratic utility function!
- All investors agree on the planning horizon and the distributions of security returns.
- There are no frictions in the capital markets.
- Note that the CAPM can be derived without assumptions 2 and 3.

CML

- From the one-fund separation theorem we know that all investors choose portfolios which are a **combination of the tangency portfolio and the risk-free asset.**





CML

- **The tangency portfolio is the same for all investors**
 - ↳ **tangency portfolio = summation of all assets = "market portfolio"**. It must contain shares of every stock in proportion to that stocks' representation in the entire market (i.e. to that stocks' market capitalization).

Market Capitalization (only three assets)					
Security	Shares outstanding	Price	Capitalization	Market weight	
Jazz, Inc.	10,000	\$6.00	60,000	15%	
Classical, Inc.	30,000	\$4.00	120,000	30%	
Rock, Inc.	40,000	\$5.50	220,000	55%	
			400,000		



CML

- In equilibrium returns of the assets have to adjust such that the market portfolio is efficient! How does this happen?
 - The return of an asset depends on its initial and its final price.
 - All investors do have the same ideas about the distribution of the final prices (by assumption).
 - Given some initial prices they solve for the best portfolios in a mean-variance sense.
 - They place orders to acquire their portfolios.
 - Now the market might clear (demand=supply) or not.
 - If it does not clear prices have to adjust (hence returns change) and investors have to find their new optimal portfolio.
 - They place orders again.
 - This procedure is reiterated until the market clears which can only happen when the market portfolio is efficient.



SML

- Theorem: If the market is efficient then there exists a perfect linear relation between the beta factors for stocks and their expected rates of return.

$$E(r_i) = r_F + \beta_i [E(r_M) - r_F], \quad \text{where } \beta_i = \frac{\text{Cov}(r_i, r_M)}{\sigma^2(r_M)}$$

The expected rate of return for a stock is the sum of

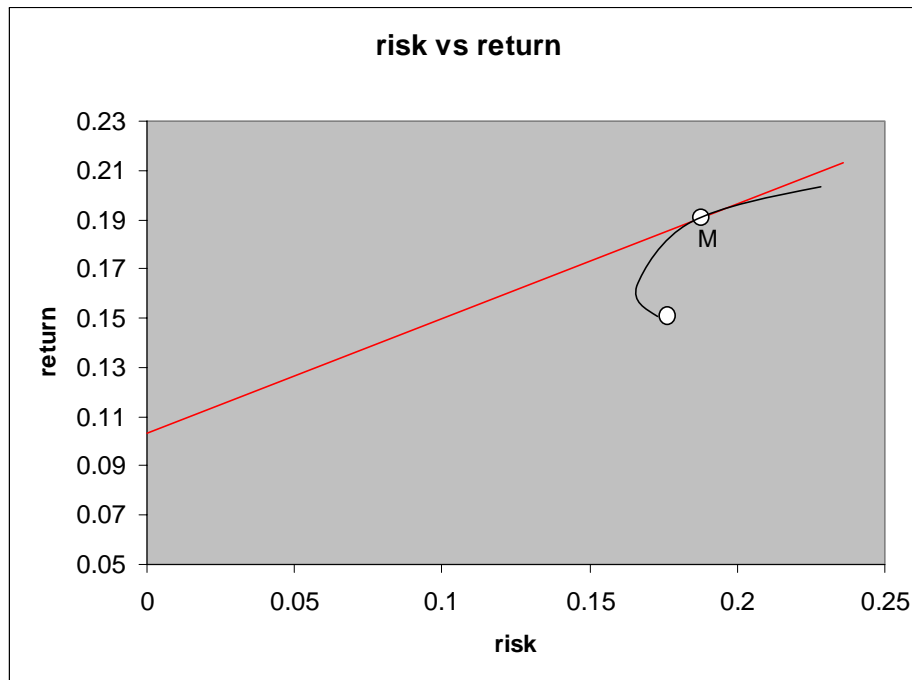
- the risk free rate (compensating for the delay in consumption) and
- the risk premium for security i (compensating for taking risk):
 - risk measure for security i and
 - market risk premium.

SML

- Proof:
 - Consider the following portfolio

$$\bar{r}_\alpha = \alpha \bar{r}_i + (1 - \alpha) \bar{r}_M$$

$$\sigma_\alpha = \sqrt{\alpha^2 \sigma_i^2 + 2\alpha(1 - \alpha)\sigma_{iM} + (1 - \alpha)^2 \sigma_M^2}$$



SML

- The curve generated by asset i and M cannot cross the capital market line, i.p. the curve at $\alpha=0$ has to be tangent to the CML, i.e.

$$\left. \frac{d\bar{r}_\alpha}{d\bar{\sigma}_\alpha} \right|_{\alpha=0} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

- To prove this we first calculate the derivative:

$$\frac{d\bar{r}_\alpha}{d\bar{\sigma}_\alpha} = \frac{d\bar{r}_\alpha / d\alpha}{d\bar{\sigma}_\alpha / d\alpha} = \frac{\bar{r}_i - \bar{r}_M}{(\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2) / \sigma_\alpha}$$

- Now we are able to evaluate this derivative at $\alpha=0$ and set it equal to the slope of the CML

$$\left. \frac{d\bar{r}_\alpha}{d\bar{\sigma}_\alpha} \right|_{\alpha=0} = \frac{\bar{r}_i - \bar{r}_M}{(\sigma_{iM} - \sigma_M^2) / \sigma_M} \quad \text{Note: } \sigma_\alpha|_{\alpha=0} = \sigma_M$$

$$\frac{\bar{r}_i - \bar{r}_M}{(\sigma_{iM} - \sigma_M^2) / \sigma_M} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

- Solving for r_i yields the desired result!

SML

- 1st conclusion

$$E(r_i) - r_F = \beta_i [E(r_M) - r_F], \quad \text{where } \beta_i = \frac{\text{Cov}(r_i, r_M)}{\sigma^2(r_M)}$$

- β_i ... beta of the asset
- $(\bar{r}_i - r_f)$... expected excess return of asset i
- The CAPM says that the **expected excess return of any asset is proportional to the expected excess return of the market portfolio!**
- Note that the expected return is independent of σ_i , i.p. **two assets with the same covariance with the market portfolio have the same expected return irrespective of their actual "risk"!**

SML

- 2nd conclusion

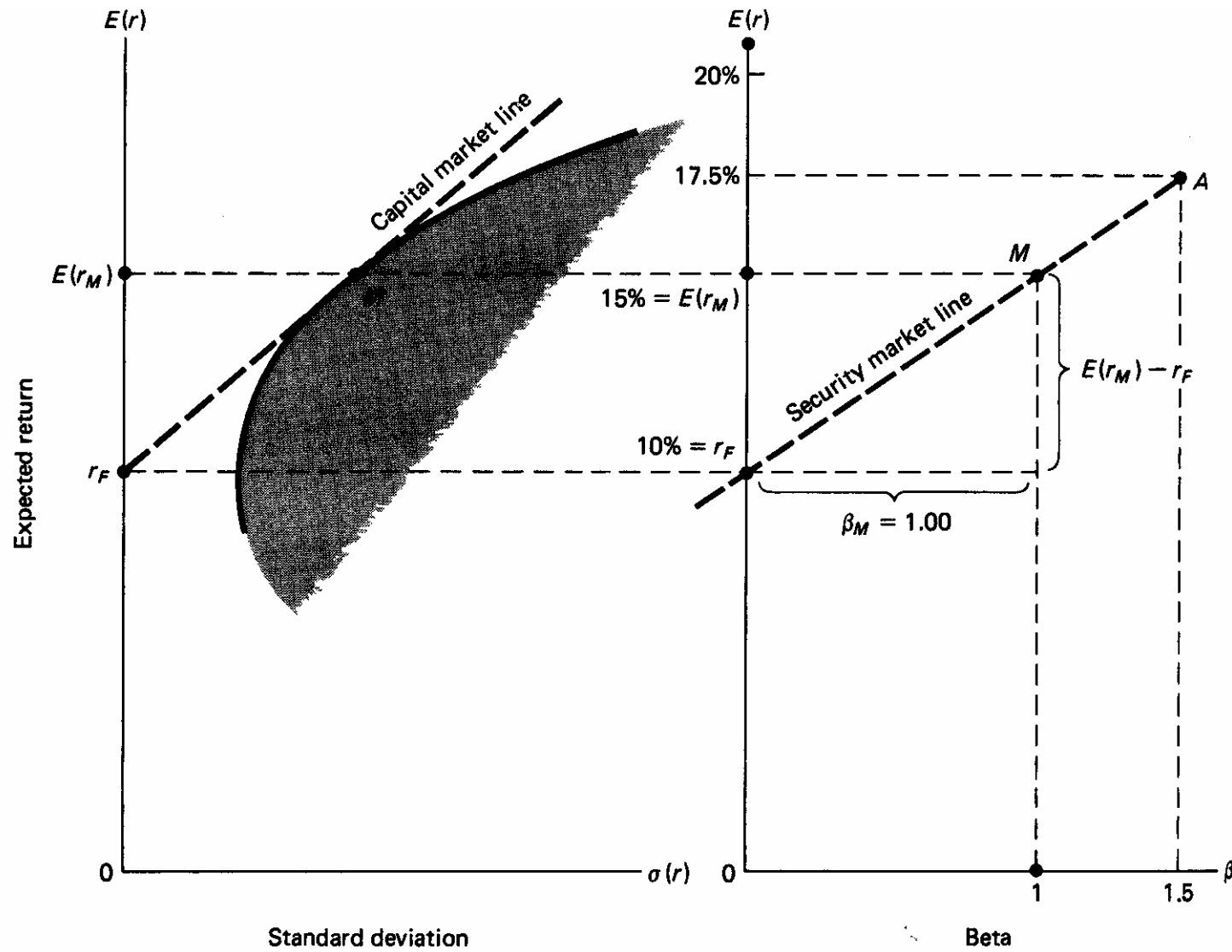
$$r_i = r_F + \beta_i [r_M - r_F] + \varepsilon_i$$

- $E(\varepsilon_i) = 0 \Leftrightarrow E(r_i) = r_F + \beta_i [E(r_M) - r_F]$
- $\text{cov}(\varepsilon_i, r_M) = 0 \Leftrightarrow \sigma_i^2 = \beta_i^2 \sigma_M^2 + \text{var}(\varepsilon_i), \text{ i.e.}$

the variance consists of two parts:

- market - / systematic risk and
- unique - / unsystematic - / diversifiable - / idiosyncratic risk.

CML and SML





Critique

- Testability of the CAPM: Roll's critique
 - If the portfolio that is used for the computation of beta lies on the minimum-variance set, then the expected returns have to lie on the SML ↻ not necessary that we use the efficient market portfolio for the computation of betas (see the proof of the SML).
 - The CAPM states that the market portfolio is efficient. Since it is impossible to observe the market portfolio this hypothesis cannot be tested.
- Empirical relevance of the CAPM

Despite Roll's critique there are still attempts to empirically verify

 - whether expected returns are a linear function of their betas with a market index and
 - whether market betas are sufficient to explain the variation of the expected returns.
 - Fama and French (1992): beta is not even significant!
 - Fama and French (1992): firm size (significant and negative), book-to-market ratio (significant and positive), leverage ratio (significant and either negative or positive), and earnings to price ratio (not significant if firm size and book-to-market equity are already included).
 - Fama and French (1993): returns are not only determined by one factor but by more.



Examples

Refer to the following data for Problems 1 and 2.

<i>Stock i</i>	<i>Correlation Coefficient i with M</i>	<i>Standard Deviation of i</i>
<i>A</i>	.5	.25
<i>B</i>	.3	.30

$$E(r_M) = .12$$

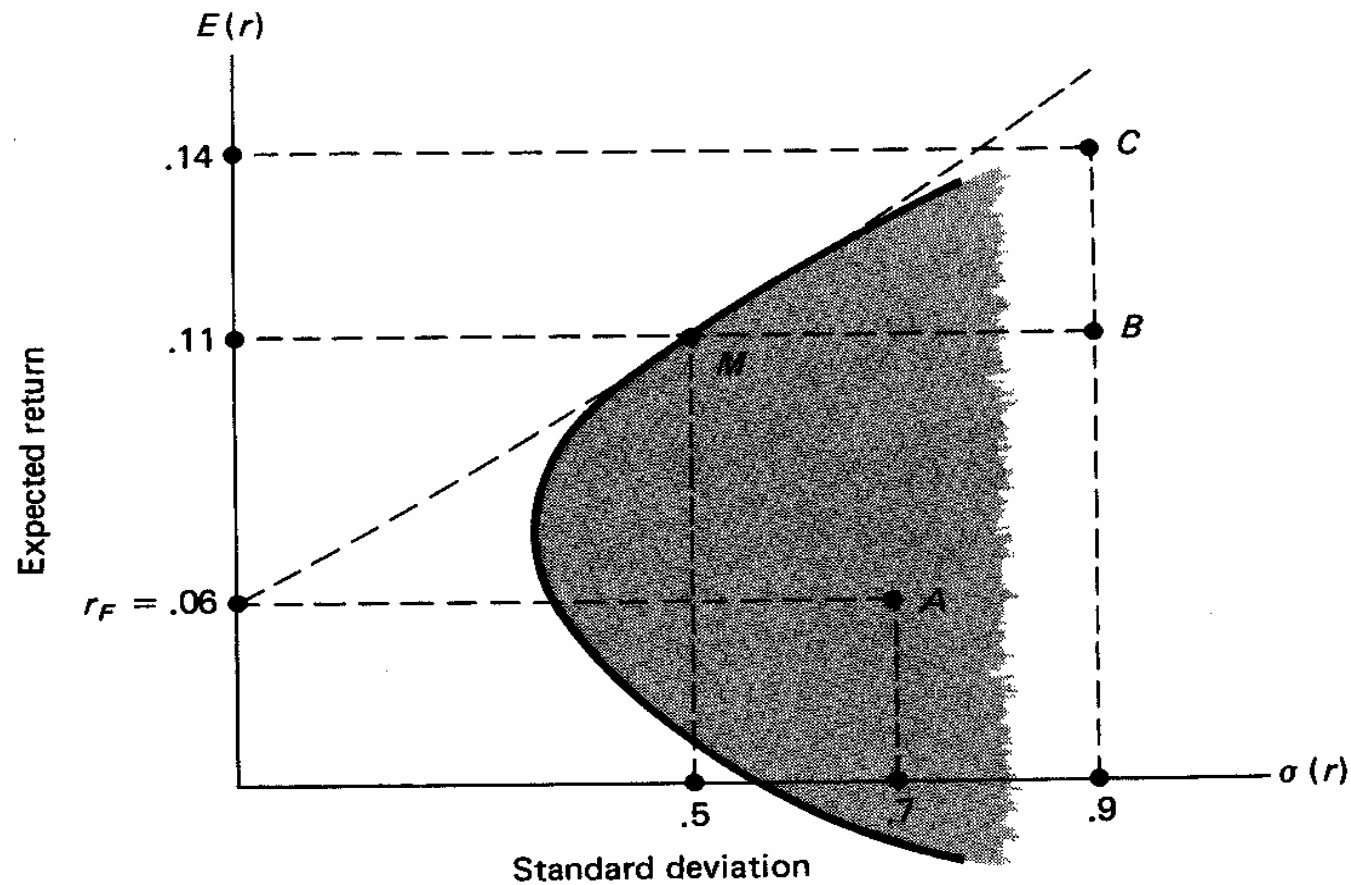
$$r_F = .05$$

$$\sigma^2(r_M) = .01$$

1. Compute betas for
 - a. Stock *A*.
 - b. Stock *B*.
 - c. For an equally weighted portfolio of stocks *A* and *B*.
2. Compute the equilibrium expected return according to the CAPM for
 - a. Stock *A*.
 - b. Stock *B*.
 - c. The portfolio indicated in Problem 1(c).

Examples

For questions 8 through 11 refer to the following diagram:





Examples

8. What are the beta values of stocks A , B , and C ?
9. What are the residual variances of stocks A , B , and C ?
10. Consider a portfolio consisting of 20 percent invested in A and 80 percent invested in C .
 - a. What is the beta of this portfolio?
 - b. According to the CAPM, what should the portfolio's equilibrium return be?
11. Evaluate the following statement: Stocks B and C should be viewed as equally risky, since they have the same standard deviation.



Performance measurement

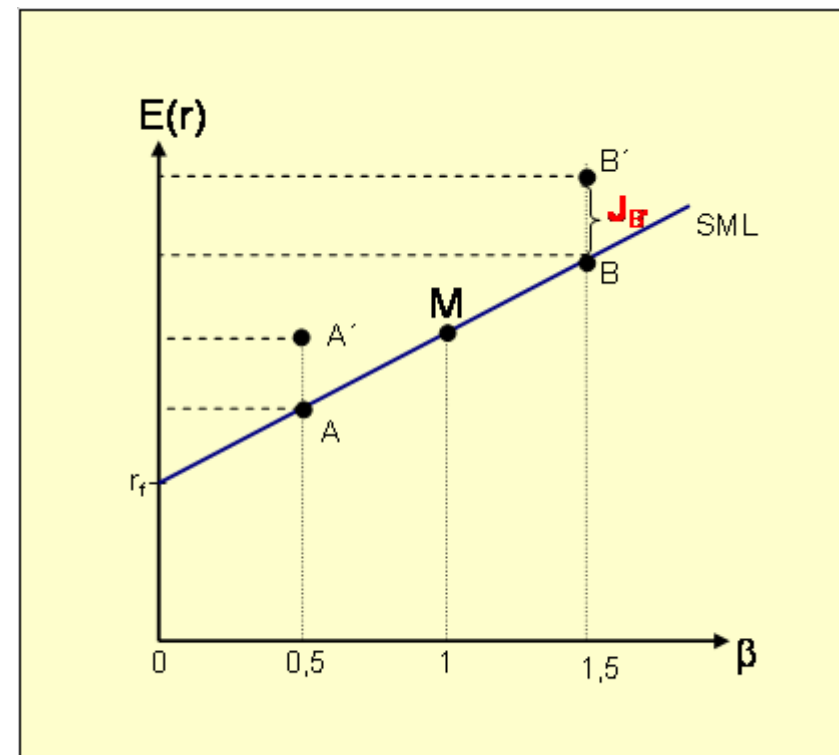
- The CAPM is often used as benchmark for portfolio performance.
- Assumption: **Pricing structure in the market is that of our standard CAPM!**
- The fund manager has done well when she beats
 - the SML (Jensen index, Treynor Index) and / or
 - the CML (Sharpe ratio, M² measure).



Jensen index

- Benchmark = SML
- **Jensen index = expected rate of return on the portfolio - what its expected return would be if the portfolio were positioned on the SML, i.e.**

$$J_P = E[r_P] - \left\{ r_f + \beta_P (E[r_m] - r_f) \right\}$$





Jensen index

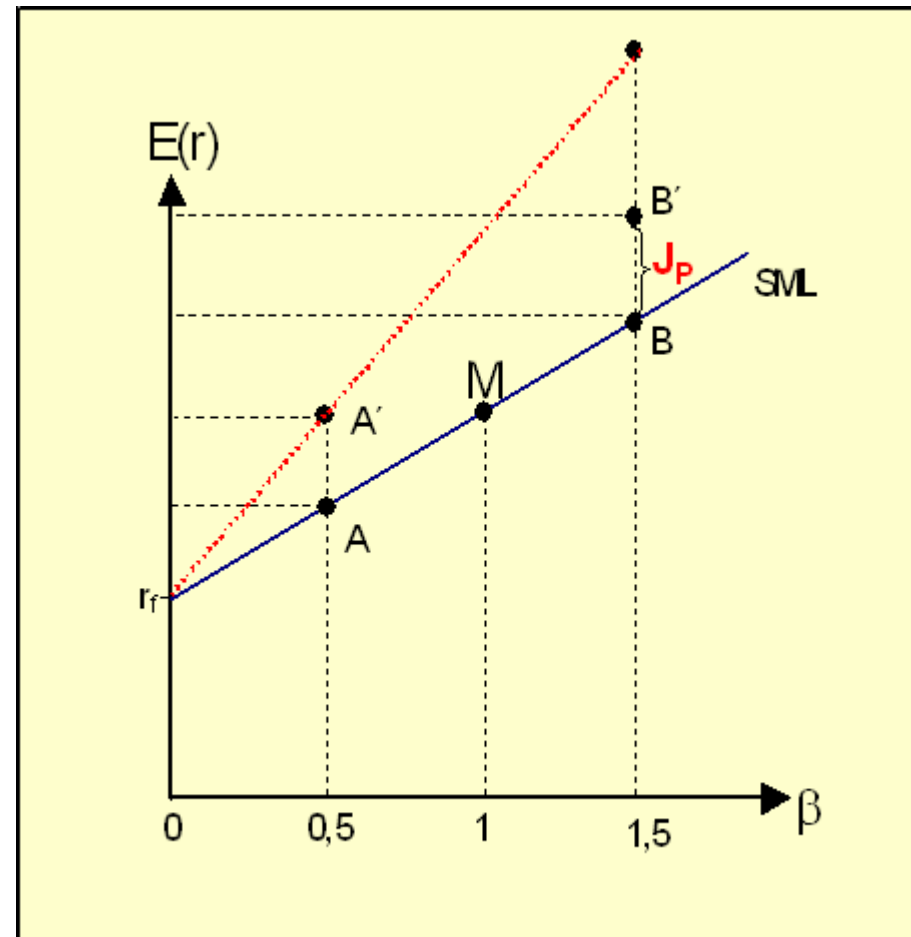
- If the fund has a positive Jensen index, it is positioned above the SML, and it is considered to have a good performance (and vice versa).
- Jensen index is sensitive to the magnitude of the excess return captured by the manager.
- Jensen index is not sensitive to the number of different securities for which a manager captured excess returns.



Treynor index

- Benchmark = SML
- **Treynor index = risk premium earned per unit of risk taken, i.e.**

$$T_P = \frac{E[r_P] - r_f}{\beta_P}$$





Treynor index

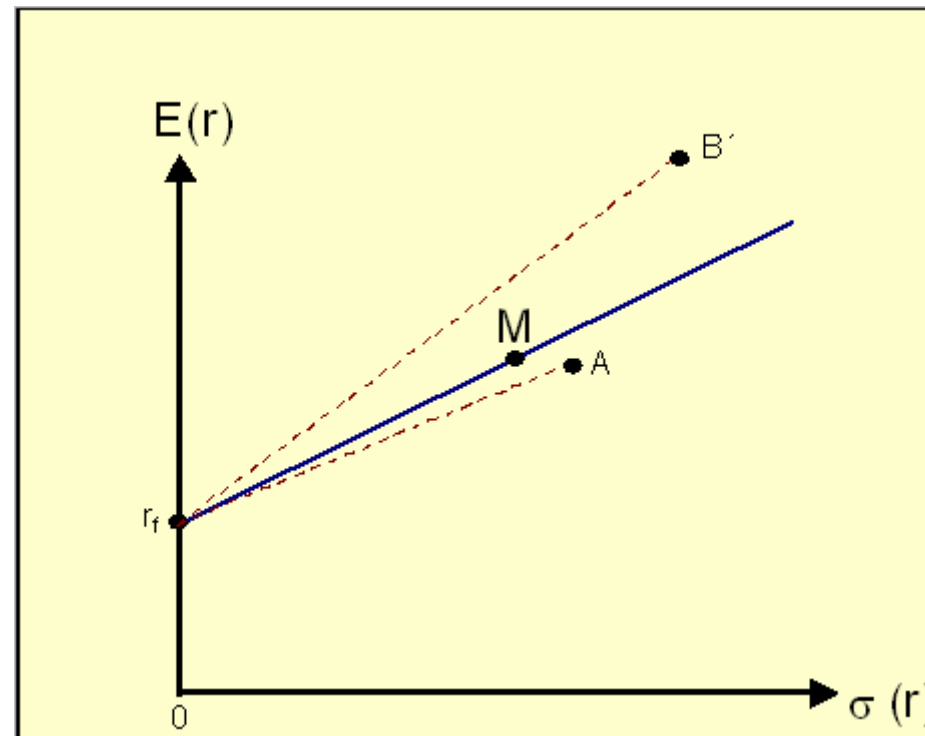
- Advantage over the Jensen index: Treynor index takes the opportunity to lever into account! Given that we can borrow at the risk-free rate we can lever a position in A' to attain a position at A^* . A^* has the same beta as B' but it has a higher expected rate of return.
- The fund with the higher Treynor index is considered to be better than the fund with a lower Treynor index.
- Treynor index is sensitive to the magnitude of the excess return captured by the manager.
- Treynor index is not sensitive to the number of different securities for which a manager captured excess returns.



Sharpe ratio

- Benchmark = CML
- **Sharpe ratio = risk premium earned per unit of risk exposure, i.e.**

$$S_P = \frac{E[r_P] - r_f}{\sigma_P}$$





Sharpe ratio

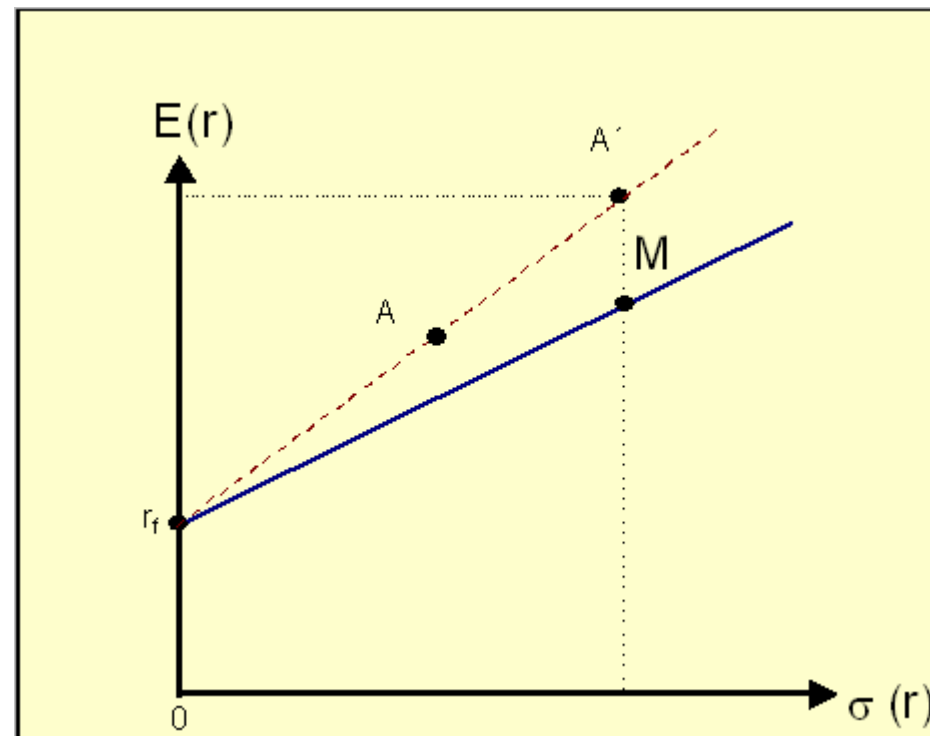
- If the fund has a higher Sharpe ratio than the market, it is positioned above the CML, and it is considered to have a good performance, i.e. the fund “outperformed the market” (and vice versa).
- Sharpe ratio is sensitive to the magnitude of the excess return captured by the manager.
- Sharpe ratio is sensitive to the number of different securities for which a manager captured excess returns.



M² Measure

- Benchmark = CML
- **M² Measure = risk-free rate of return + risk premium earned per unit of risk exposure * vola of the market, i.e.**

$$M_P = r_f + \frac{E[r_P] - r_f}{\sigma_P} \sigma_M$$





M² Measure

- M² measure takes the opportunity to lever into account! Given that we can borrow at the risk-free rate we can lever a position in A to attain a position at A'. A' has the same beta as the market but it has a higher expected rate of return.
- The fund with the higher M² measure is considered to be better than the fund with a lower M² measure.
- M² measure is sensitive to the magnitude of the excess return captured by the manager.
- M² measure is sensitive to the number of different securities for which a manager captured excess returns.

Overview

- Motivation
- Single-factor models (SFM)
- Multi-factor models (MFM)
- Arbitrage Pricing Theory (APT)
- Additional literature
 - Haugen, R. A., *Modern Investment Theory*, 5th edition, 2001.
Relevant chapters: 6 and 10.

Motivation

- Estimates of expected returns and covariances between the securities ↪ compute efficient set
- How can one get these estimates?
 - **Sampling from past returns** ↪ (arithmetic) mean and sample covariance
 - Advantage: easy and fast
 - Disadvantage: sample size * ↪ sampling error *
 - BUT sample size * ↪ probability that series of stock returns doesn't reflect the contemporary character of the firm ***

Motivation

- **Factor models**

- **Risk factors (rate of inflation, growth in industrial production, ...)** induce the stock prices to go up and down from period to period

Different stocks respond to movements in the risk factors to different degrees ↪ different **future covariances** of return between different stocks

- **Expected return factors (firm characteristics, e.g. firm size, liquidity, ...)** ↪ explanations why some firms produce higher returns, on average, than others

Assumption

- Security returns are correlated for only one reason, i.p. **each security is assumed to respond to the pull of a single factor**, which is usually taken to be the market portfolio!

$$\Rightarrow \text{Cov}(r_J, r_K) = \beta_{J,F1} \beta_{K,F1} \sigma_{F1}^2$$

$$r_{J,t} = A_J + \beta_{J,F1} r_{F1,t} + \varepsilon_{J,t}, \quad F1 = \text{Market}$$

- **Implicit assumption:** 2 types of events produce the period-to-period variability in a stock's rate of return:
 - **Macro events ... affect nearly all firms** \rightarrow change in r_M \rightarrow change in rates of return on individual securities (e.g. unexpected change in the rate of inflation, change in the Federal Reserve discount rate, ...)
 - **Micro events ... affect only individual firms**, i.e. they are assumed to have no effects on other firms and they have no effect on r_M \rightarrow cause the appearance of residuals or deviations from the characteristic line; **residuals of different companies are uncorrelated with each other:**

$$\text{Cov}(\varepsilon_J, \varepsilon_K) = 0.$$

Variance

- **Total variance of the return on a security**

$$\Rightarrow \sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon_i}^2$$

- **Portfolio variance**

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma_{\varepsilon_P}^2$$

$$\text{where } \beta_P = \sum_{J=1}^N x_J * \beta_J$$

$$\text{and due to } \text{cov}(\varepsilon_J, \varepsilon_K) = 0 \Rightarrow \sigma_{\varepsilon_P}^2 = \sum_{J=1}^N x_J^2 * \sigma_{\varepsilon_J}^2$$



SFM variance vs. Markowitz variance

- **Markowitz formula is perfectly accurate, given the accuracy of the covariance estimates.** (The Markowitz model makes no assumptions regarding the process generating security returns.)
- The SFM assumes the residuals are uncorrelated across different companies ↻ **SFM variance is only an approximation of the true variance**, it is only as accurate as our assumption regarding the residuals!

Examples

- Given the following info and the assumption of a SFM, what is the covariance between stocks A and B?

$$\beta_A = 0,85; \quad \beta_B = 1,3; \quad \sigma_{F1=M}^2 = 0,09.$$

- Consider a portfolio of stock A and B, where the weight of stock A is 2/3 and assume the following:

$$\sigma_{\varepsilon_A}^2 = 0,02; \quad \sigma_{\varepsilon_B}^2 = 0,06; \quad \text{Cov}(\varepsilon_A, \varepsilon_B) = 0,01.$$

- What is the residual variance of the portfolio if the SFM is assumed?
- What is the residual variance of the portfolio without the SFM?

Example

- Consider a portfolio of stock A and B, where the weight of stock A is 1/2 and assume the following:

$$\beta_A = 0,5; \quad \beta_B = 1,5; \quad \sigma_{\varepsilon_A}^2 = 0,04; \quad \sigma_{\varepsilon_B}^2 = 0,08;$$

$$\sigma_A^2 = 0,0625; \quad \sigma_B^2 = 0,2825.$$

- What is the beta coefficient of the portfolio?
- Compute the residual variance of the portfolio assuming a SFM model.
- Compute the variance of the portfolio assuming a SFM model.

Example

- Consider a portfolio of stock A and B, where the weight of stock A is 0,4 and assume the following:

$$\rho_{A,B} = 0,5; \quad \sigma_M = 0,1; \quad \rho_{A,M} = 0; \quad \rho_{B,M} = 0,5;$$

$$\sigma_A = 0,1; \quad \sigma_B = 0,2.$$

- What are the beta values for stock A and B?
- What is the covariance between stock A and B, assuming a SFM model?
- What is the true covariance between stock A and B?
- What is the beta for the portfolio?
- What is the variance of the portfolio, assuming a SFM model?
- What is the true variance of the portfolio?

Assumption

- Security returns are not longer correlated for only one reason, i.p. **each security is assumed to respond to the pull of two or more factors!**

$$\Rightarrow \text{Cov}(r_J, r_K) = \beta_{J,F1}\beta_{K,F1}\sigma_{F1}^2 + \beta_{J,F2}\beta_{K,F2}\sigma_{F2}^2 + \dots$$

$$r_{J,t} = A_J + \beta_{J,F1}r_{F1,t} + \beta_{J,F2}r_{F2,t} + \dots + \varepsilon_{J,t}$$

- $\text{cov}(F1, F2) = 0$ and

$$\text{cov}(\varepsilon_J, \varepsilon_K) = 0$$

Variance

- **Total variance of the return on a security**

$$\Rightarrow \sigma_i^2 = \beta_{i,F1}^2 \sigma_{F1}^2 + \beta_{i,F2}^2 \sigma_{F2}^2 + \dots + \sigma_{\varepsilon_i}^2$$

- **Portfolio variance**

$$\sigma_P^2 = \beta_{P,F1}^2 \sigma_{F1}^2 + \beta_{P,F2}^2 \sigma_{F2}^2 + \dots + \sigma_{\varepsilon_P}^2$$

where $\beta_{P,Fi} = \sum_{J=1}^N x_J * \beta_{J,Fi}$ for all i

and due to $\text{cov}(\varepsilon_J, \varepsilon_K) = 0 \Rightarrow \sigma_{\varepsilon_P}^2 = \sum_{J=1}^N x_J^2 * \sigma_{\varepsilon_J}^2$



Example

- A 2-factor model is being employed, one a market factor (M) and the other a factor of unexpected changes in the growth of industrial production (g).

$$\beta_{A,M} = 0,6; \quad \beta_{B,M} = 0,9; \quad \beta_{A,g} = 0,2; \quad \beta_{B,g} = 0,1;$$

$$\sigma_{\varepsilon_A}^2 = 0,05; \quad \sigma_{\varepsilon_B}^2 = 0,02; \quad \sigma_M^2 = 0,12; \quad \sigma_g^2 = 0,1;$$

$$\text{cov}(\varepsilon_A, \varepsilon_B) = 0,02; \quad \text{cov}(M, g) = 0.$$

- Compute the variance of stock A and B.
- Compute the market and growth beta for an equally weighted portfolio of stock A and B.
- Assume you had constructed an equally weighted portfolio of stocks A and B. Compute the residual variance and the variance of this portfolio in two ways:
 - making the simplifying assumption of the 2-factor model about residual covariance and
 - without making this assumption.

Motivation

- Problems associated with the CAPM ↻ growing interest in alternative valuation models!
- Most important alternative = **APT**; first introduced by Ross (1976)
- Major advantages over the CAPM
 - APT needs **no specific assumptions about the utility functions** of the decision maker (investor); i.p. the APT requires that bounds be placed on investors' utility functions, but the bounds are less restrictive
 - The **model can be tested empirically**

Assumptions

- **Securities' returns can be described through an index or factor model**

$$r_{J,t} = A_J + \beta_{J,F1}r_{F1,t} + \beta_{J,F2}r_{F2,t} + \dots + \varepsilon_{J,t}$$

$$\sigma_P^2 = \beta_{P,F1}^2 \sigma_{F1}^2 + \beta_{P,F2}^2 \sigma_{F2}^2 + \dots + \sigma_{\varepsilon_P}^2$$

$$\text{where } \beta_{P,Fi} = \sum_{J=1}^N x_J * \beta_{J,Fi} \text{ for all } i$$

$$\text{and } \sigma_{\varepsilon_P}^2 = \sum_{J=1}^N x_J^2 * \sigma_{\varepsilon_J}^2$$

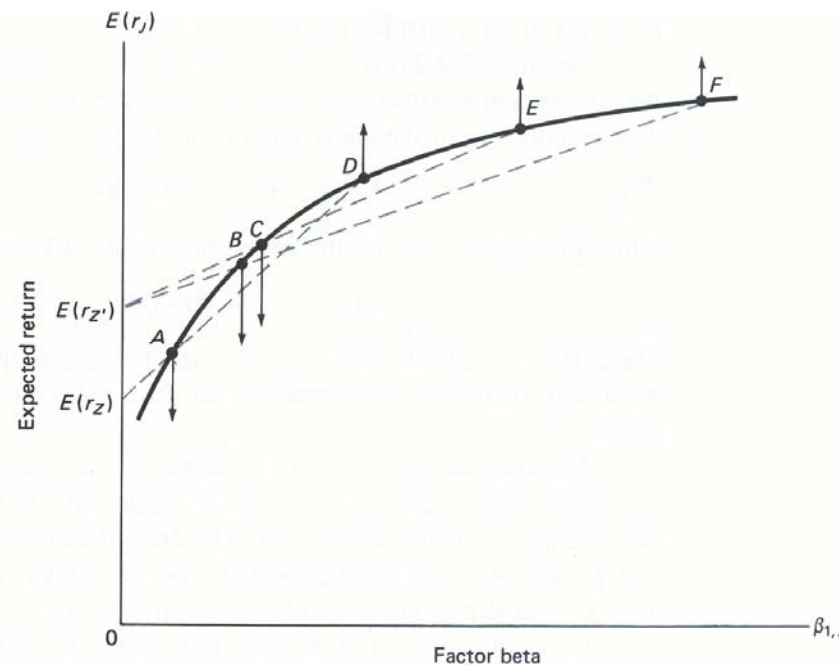
The implicit assumption that the covariance between factors is equal to zero is not a necessary assumption.

- **Large amounts of securities** (many more securities than number of factors) **and the possibility of short-selling**



Expected return and risk relationship

- Given these assumptions we “derive” the approximate relationship between expected return and risk under the APT
- Suppose a single factor can explain all the covariance that exist between stocks, i.e. **single-factor APT**
 - What will the relationship between $E(r_j)$ and $\beta_{j,F1}$ look like?
 - Suppose it looks as follows:





Expected return and risk relationship

- There are an unlimited number of securities along the curved line; six of these securities are labeled A, B, C, D, E, and F
- Portfolio beta and expected portfolio return are simple weighted averages
 - ↳ combination lines can be drawn as straight lines passing through the points on the graph
- Sell E short and use the proceeds to invest in C ↳ we can create a zero-beta portfolio $E(r_{Z'})$
- Sell E and F short and use the proceeds to invest in C and B ↳ we can create the same zero-beta portfolio $E(r_{Z'})$
- We can create $E(r_{Z'})$ by using as many pairs of stocks as we want, i.p. we can use an infinite number of pairs ↳ $\beta_{Z'}$ is zero by construction and $\sigma_{\varepsilon_p}^2 = \sum_{j=1}^{N=\infty} x_j^2 * \sigma_{\varepsilon_j}^2 \approx 0$ ↳ portfolio has approximately zero total variance
- Construct a portfolio positioned at $E(r_Z)$
- **Sell short $E(r_Z)$ and use the proceeds to invest in $E(r_{Z'})$ ↳ arbitrage profit**

Expected return and risk relationship

- We'll all be trying to take advantage of this arbitrage opportunity, selling short stocks like D, E, and F while buying stocks such as A, B, and C
 - **selling short stocks like D, E, and F**
 - ↳ prices *
 - ↳ expected rates of return *
 - **buying stocks such as A, B, and C**
 - ↳ prices *
 - ↳ expected rates of return *
 - **The effect of all this will be to “unbend” the line until the general relationship is approximately linear**

$$E[r_J] \approx E[r_Z] + \lambda_{F1} \beta_{J,F1}$$

$$E[r_P] \approx E[r_Z] + \lambda_{F1} \beta_{P,F1}$$

$$E[r_J] \approx E[r_Z] + \sum_{i=1}^n \lambda_{Fi} \beta_{J,Fi}$$

$$E[r_P] \approx E[r_Z] + \sum_{i=1}^n \lambda_{Fi} \beta_{P,Fi}$$

Important facts

- Given factor prices s.t. there exist a linear relationship between the betas with reference to the market portfolio and expected rates of return
↳ **CAPM and APT are completely consistent** (BUT CAPM is not a special case of the APT, since the CAPM assumes nothing about the structure of security returns other than that possibly they are normal distributed. Normal distributions, however, do not necessarily imply the linear factor structure required by the APT).
- The APT is completely silent with respect to what the factors stand for!
- Chen, Roll, Ross (1986):
 - Inflation
 - Industrial production
 - Risk premiums
 - Term structures

Example

- Assume that a 3-factor APT model is appropriate. The expected return on a portfolio with zero beta values is 5%. You are interested in an equally weighted portfolio of 2 stocks, A and B. You should compute the approximate expected return on the portfolio, given the following info.

$$\beta_{A,F1} = 0,3; \quad \beta_{B,F1} = 0,5;$$

$$\beta_{A,F2} = 0,2; \quad \beta_{B,F2} = 0,6;$$

$$\beta_{A,F3} = 1; \quad \beta_{B,F3} = 0,7;$$

$$\lambda_{F1} = 0,07; \quad \lambda_{F2} = 0,09; \quad \lambda_{F3} = 0,02.$$

Example

- Assume a 1-factor APT model having the expected return-beta relationship as graphed. Find a portfolio of A and C that would result in a beta of zero. What is the expected return on this portfolio?

Overview

- Fixed income (FI) securities
- Floating rate notes (FRNs)
- Additional literature
 - Haugen, R. A., *Modern Investment Theory*, 5th edition, 2001.
Relevant chapters: 2 and 15.

Overview

- Introduction
 - Equity vs. debt instruments
 - Definition
 - Characteristics
 - Bond Markets

- Valuation

- Credit risk

- Market risk

Equity vs. debt instruments

- Equity instruments
 - Residual claim (on the earnings of the firm)
 - Control in non-default states (equity > 0)
 - Dividend * (1-corporate tax rate)
- Debt instruments
 - Contracted claim ↻ “fixed income securities”
 - Control in default states (equity < 0)
 - Tax deductibility of interest



Definition

- Bonds are called fixed income securities, because
 - a fixed amount, **“face value” (FV) / “principal” / “par value”**, is repaid at the date of maturity and
 - a fixed amount, **“coupon” (c) / “interest”** is paid periodically.

- A bond is a security that **obligates the issuer to make specified interest and principal payments to the holder on specified dates**,
 - **$t = 0$: bondholder pays the price / present value (P),**
 - **$0 < t \leq T$: bondholder (ev.) receives coupon payments, and**
 - **$t = T$: bondholder receives the face value.**



Characteristics

The main aspects that can be set in a bond contract:

- **Face value (FV) / principal / par value:** original value due at maturity!
It is not the same as the market value!
- **Coupon (c) / interest:** income that investor will receive over the life of the issue.



Characteristics

- **Zero-coupon bonds / pure discount bonds:** pay no coupons prior to maturity and pay the face value at maturity ↻ single payment at maturity!
- **Fixed**
 - **(Straight) coupon bonds / bullet bonds:** pay a stated coupon periodically and pay the face value at maturity
 - **Consol bonds / perpetual bonds:** no maturity date and pay a stated coupon periodically ↻ they pay only interest
 - **Annuity bonds:** pay a mix of interest and principal for a finite amount of time
 - **Deferred coupon bonds:** permit the issuer to avoid coupon payments for a certain period of time
- **Floating ↻ "floating rate note" (FRN):** coupon is reset periodically

Characteristics

- **Maturity (T):** date when (or time until) the bond expires (or matures); maximum length of time the borrower has to pay off the principal in full.
 - **Bill / paper / short-term issues: $T \leq 1y$**
 - **Notes / intermediate-term issues: $1y < T \leq 10y$**
 - **Bonds / long-term obligations: $T > 10y$**
- Type of ownership
 - Bearer bond
 - Registered bond



Characteristics

- **Bond covenants / “me first rules”:** help to protect bondholders from the moral hazard of equity holders!
 - **Asset covenants**
 - **Dividend covenant:** restricts the payment of dividends
 - Secured / **senior bonds:** asset backed, e.g. mortgage bonds
 - **Unsecured bonds**
 - Subordinated / **junior bonds:** claim that is subordinated to other debt instruments; lowest priority
 - **Sinking fund covenants:** bond must be paid off systematically over its life, i.e. a part of the principal must be repaid before T, e.g. annuity bond

Characteristics

- **Financial ratio covenants:** if a target ratio is not met the firm is technically in default, e.g. working capital ($= \text{current A} - \text{current L}$) $> x$, interest coverage ratio ($= \text{earnings} / \text{interest}$) $> x$, ...
- **Financing covenants:** description of the amount of additional debt the firm may issue and the claims to assets that this additional debt might have in the event of default

Characteristics

- Embedded options: options included in bond contracts
 - NOTE: **The embedded options have a price!**
 - Freely ...: option can be exercised at any time (notification period)
 - Deferred ... provision: option cannot be exercised during a certain period
- **Callable bond:** allows the issuing firm to retire the bonds before T by paying a pre-specified price (lower price)
- **Putable bond:** allows the bondholder to sell the bond back to the issuer (higher price); e.g. poison put: allows the bondholder to sell the bond back to the issuer in the case that someone obtains more than 50% of the company
- **Convertible**
- **Exchangeability**
- **Embedded currency option**

Bond markets

- Domestic bond market
 - Most issues are listed at an exchange
 - BUT most trading takes place over-the-counter (OTC)
- Foreign bond market
 - **Foreign bonds = issued by foreign borrowers in a nation's domestic market and denominated in the nation's domestic currency**
 - Yankee-, Samurai-, Brady-Bonds, ...
- Eurobond market
 - **Eurobonds = denominated in a particular currency and issued simultaneously in the markets of several nations**
 - Traded exclusively over-the-counter (OTC)

Bond markets

- The 5 most important markets for bonds

	Total value [billion USD], 1997
United States	11,218
Japan	4,173
Germany	2,943
Italy	1,271
United Kingdom	856

Bond markets

- Distribution of bonds according to type of issuer, 1997

	US [%]	Japan [%]	BRD [%]
Government	25	43	23
Federal (agency), government related	25	16	3
State, local, municipal	10	6	15
Bank		18	38
Corporate	26	10	0,1
International	14	7	21



Valuation

... simply compute the **present value of the promised future CF streams** (effective periodic interest rate = simple p.a. interest rate / compounding period)

Examples

- What is the current market price of a US Treasury strip that matures in exactly 5 years and has a face value of \$ 1000, using an annual interest rate of $R=7.5\%$ (annual compounding)?
- What is the annual interest rate on a US Treasury strip that pays \$1000 in exactly 7 years and is currently selling for \$ 591.11 (annual compounding)?
- What is the market price of a US Treasury bond that has a coupon rate of 9% p.a., a face value of \$1000, and matures exactly 2 years from today if the interest rate is 10% compounded annually?



Examples

- What is the market price of a US Treasury bond that has a coupon rate of 9% p.a., a face value of \$1000 and matures exactly 2 years from today if the interest rate is 10% p.a. compounded semi-annually?
- What is the market price of a US Treasury bond that has a coupon rate of 9% p.a., a face value of \$1000 and matures exactly 10 years from today if the interest rate is 10% p.a. compounded semi-annually?
- What is the market price of a consol bond that has a coupon rate of 9% p.a. if the interest rate is 10% p.a. compounded semi-annually?

$$PV = \frac{c}{i} \left[1 - \frac{1}{(1+i)^n} \right] + \frac{FV}{(1+i)^n} = FV [1 + (c-i) * Ann], \quad \text{where } Ann = \frac{1 - (1+i)^{-n}}{i}$$

$$PV = \frac{c}{i}$$



Valuation

- **Yield-to-maturity (y)** / internal rate of return (IRR): (constant) discount rate that makes the discounted value of the promised future CFs equal to the market price of the bond (... forecast of the average annual rate of return, under the assumption that the coupon can be reinvested at this rate)
- **Zero rate (z_t)** / spot rate: internal rate of return of a zero-coupon bond

$$PV = \sum_{t=1}^T \frac{c}{(1+y)^t} + \frac{FV}{(1+y)^T} = \sum_{t=1}^T \frac{c}{(1+z_t)^t} + \frac{FV}{(1+z_t)^T}$$

$$\text{if } c = 0 \Rightarrow y = z_t$$

$$\text{if } T = 1 \text{ and number of CFs per year} = 1 \Rightarrow y = z_1$$

Quotes

- Selling **at par**: $PV = FV$
- Selling **at a discount**: $PV < FV$
- Selling **at a premium**: $PV > FV$
- *What can you infer about the relationship of the yield and the coupon?*
- $PV = \text{quoted price} + \text{accrued interest}$
 - Accrued interest = c [p.a.] * time since last c [y]
 - Time since last c [y] ... usually actual/365

Rating

- ... **expected return on a loan** 🕒 **agreed interest rate**
- Measurement of credit risk: **ratings**; a bond rating is a quality ranking of a specific debt issue
- How rating agencies rate the bonds:
 - 2 aspects of credit risk: **default probability and expected loss in the event of default**
 - Main factors taken into account: **macroeconomic data** (rule of thumb: country rating > bond rating), **industry/regulatory trends, management quality, operating/financial positions, company structure, and issue structure**

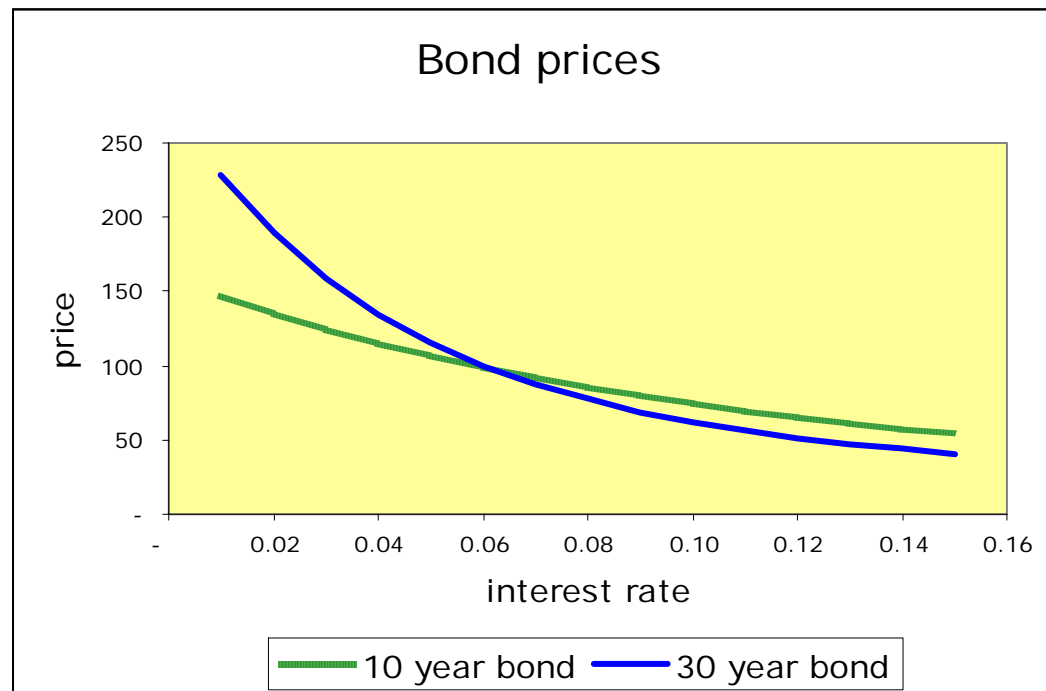
Rating

S&P	Moody's	Definition	Cum. default prob.	
			1 year	10 years
AAA	Aaa	High Grade	0.00%	1.40%
AA	Aa		0.00%	1.29%
A	A	Medium Grade	0.06%	2.17%
BBB	Baa		0.18%	4.34%
BB	Ba	Speculative	1.06%	17.73%
B	B		5.20%	29.02%
CCC	Caa	Default	19.79%	45.10%
CC	Ca			

- Rating review: upgrade, downgrade (to BB+ / Ba1 or lower "fallen angel")
- Junk bonds / high yield bonds** ... bonds with a lower rating than BBB- / Baa3 (Usually pension funds and other financial institutions are not allowed to invest in junk bonds)

Market risk

- ... risk about the uncertainty of interest rate changes
 - If interest rates rise, the price of bonds will fall
 - If interest rates fall, the price of bonds will rise
- The reaction of long-term bonds is more pronounced than the reaction of short-term bonds



Taylor series expansion

▪ Risk indications for bonds

- Taylor series \Rightarrow approximate price change, given changes in the YTM

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2!} f''(x)\Delta x^2 + \dots + \frac{1}{n!} f^n(x)\Delta x^n$$

$$f(x) = P(y)$$

$$P(y + \Delta y) = P(y) + \frac{\partial P(y)}{\partial y} \Delta y + \frac{1}{2} \frac{\partial^2 P(y)}{\partial y^2} \Delta y^2 + \dots \quad / - P(y) \quad / : P(y)$$

$$\frac{P(y + \Delta y) - P(y)}{P(y)} = \frac{\frac{\partial P(y)}{\partial y}}{P(y)} \Delta y + \frac{1}{2} \frac{\frac{\partial^2 P(y)}{\partial y^2}}{P(y)} \Delta y^2 + \dots$$

$$\frac{\Delta P}{P} = -D^M * \Delta y + \frac{1}{2} * CX * \Delta y^2 + \dots$$

Taylor series expansion

- **Duration ... how sensitive the price is to changes in the yield; %o-change in the price for a given small change in the YTM**

$$D = - \frac{\partial P}{\partial y} \frac{1+y}{P}$$

Duration for a (straight) coupon / bullet bond

$$D = \sum_{t=1}^T \frac{t * PV(CF_t)}{k * P_o}$$

k ... number of periods (CFs) per year

P ... PV of the bond

if $c=0 \Rightarrow D=T$

if $T=1$ and $k=1 \Rightarrow D=T=1$

- **Modified duration**

$$D^M \equiv \frac{D}{1+y}; \quad D = - \frac{\partial P}{\partial y} \frac{1+y}{P} \Rightarrow D^M \equiv \frac{D}{1+y} = - \frac{\partial P}{P}$$

Taylor series expansion

- **Convexity**

$$CX = \frac{\frac{\partial^2 P(y)}{\partial y^2}}{P(y)}$$

Duration for a (straight) coupon / bullet bond

$$CX = \frac{1}{\left(1 + \frac{y}{k}\right)^2} \sum_{t=1}^T \frac{t * (t+1) * PV(CF_t)}{k^2 * P_0}$$

k ... number of periods (CFs) per year

P ... PV of the bond

Overview

- Definition
- Perfectly indexed FRNs

Definition

- Coupon is not fixed but floating, i.e. **coupon is reset periodically**
 - **Reference rate: money market rate (LIBOR) or capital market rate (CMT)**; The maturity of the reference rate corresponds to the length of the coupon/reset period (... natural time lag)
 - **Reset date**: in advance (usually)
 - **Additional features: cap** (upside limited, e.g. $c = \min(\text{LIBOR}, 7\%)$); **floor** (downside limited, e.g. $c = \max(\text{LIBOR}, 2\%)$), **margin** (changed by a fixed amount, e.g. $c = \text{LIBOR} + 50\text{bp}$)

Perfectly indexed FRNs

- **Perfectly indexed FRN** = a floater **indexed to a MMT rate** with **natural time lag** which is **reset in advance** and has **no additional features**

$$PV_{t=\text{reset date}} = FV \quad (\text{i.p. } PV_{t=0} = FV)$$

$$\Rightarrow PV_t = FV \frac{1 + r_{tr=0,Tr}}{1 + r_{t,Tr}}; \quad tr = 0 < t < Tr$$



Examples

- Some time ago, a German company issued a EUR 5m FRN (at the p.a. EUR LIBOR). Suppose the current conditions are as follows:
 - The note matures in 2,5 years.
 - The EUR LIBOR rate, set 6 months ago for the current period, is 6% p.a.
 - The current six-month EUR LIBOR rate is 5% p.a.

What is the PV of the FRN?

- How would you price the FRN from above given that the coupon = EUR LIBOR + 10bp?

Overview

- Introduction
- Forwards
- Futures
- Options
- Swaps
- Additional literature
 - Sercu, P. and Uppal, R., *International Financial Markets and the Firm*, 1995. Relevant chapters: 1-8 and 10.
 - Hull, J.C., *Options, Futures, and Other Derivatives*, 5th edition, 2002. Relevant chapters: 1-4, 6, 8-12, 14, 15, and 19.
 - Shreve, S.E., *Stochastic Calculus for Finance I The Binomial Asset Pricing Model*, 2004. Relevant chapters: 1 and 2.



Definition of derivative securities

- **A derivative is a contract to buy or sell something in the future, namely the underlying.**
- **All contract details** (such as the price, the quantity to be bought or sold, the maturity, etc.) **are fixed at the time you enter the contract.**
- **The price of the derivative depends on the underlying.**
- The underlying can be everything as long as it is clearly defined!
Examples:
 - Financial prices: stocks, bonds, stock-indices, exchange rates
 - Commodities: oil price, gold, copper, coffee, orange juice concentrate, wine, energy, weather

In most cases the underlying is the price of a traded asset!

Types of derivatives

- **Forwards**
- **Futures**
- **Options**
- **Swaps**
- **Advanced products**
 - Structured products: combinations of forwards, swaps, and options
 - Hybrid debt: straight debt with embedded derivative instruments
 - Exotic options
 - Real options
 - ...

Derivatives markets

- **Over-the-counter markets**
 - **non-standardized products; tailor made contracts**
 - telephone or electronic trading
 - relative high transaction cost
 - relative high credit risk/default risk
 - ↳ illiquidity

- **Exchange-traded markets**
 - **standardized products**
 - trading floor (open outcry system) or electronic trading
 - relative low transaction cost
 - virtually no credit risk/default risk
 - ↳ liquidity

Purpose of derivatives

- **Risk management:** to hedge risks
- **Arbitrage:** to lock in arbitrage profits
- **Speculation**

Overview

- Introduction
 - Definition
 - Payoffs
 - Present values
- Pricing
- Hedging
- Problems



Definition

- A **forward**/outright **contract** is an agreement for the purchase (**long position, forward purchase, FP**) or sale (**short position, forward sale, FS**) of a **prespecified number of units of the underlying** at a certain time in the future, T , at a **prespecified price (strike price/delivery price/forward price)**, $F_{t_0, T}$.
- Note: Delivery and payment takes place in the future on a date stated in the contract, T .
- Note: It can be contrasted with a spot contract which is an agreement to buy or sell immediately.
- OTC

Payoffs

- **FP ... the holder is obligated to buy an asset worth S_T**

for $F_{t_0,T}$: $CF_T = S_T - F_{t_0,T}$

- **FS ... the holder is obligated to sell an asset worth S_T**

for $F_{t_0,T}$: $CF_T = F_{t_0,T} - S_T$



Present value for a FP

- **Suppose you want to close out a FP at time t , $t_0 < t < T$.**
- At time t : close out a FP by adding a FS for the same T
- At time T :
 - Payoff from FP: $S_T - F_{t_0, T}$
You will buy one unit of the underlying and pay $F_{t_0, T}$ for it.
 - Payoff from FS: $F_{t, T} - S_T$
You will sell one unit of the underlying and receive $F_{t, T}$ for it.
 - Sum of the payoffs: $S_T - F_{t_0, T} + F_{t, T} - S_T = F_{t, T} - F_{t_0, T}$
- Discounting with the risk-free rate gives the present value for a FP:

$$PV_t = (F_{t, T} - F_{t_0, T})e^{-rT}$$

- Note: $PV_{t_0} = (F_{t_0, T} - F_{t_0, T})e^{-rT} = 0$

133
$$PV_T = F_{T, T} - F_{t_0, T} = S_T - F_{t_0, T} = CF_T$$



Present value for a FS

- **Suppose you want to close out a FS at time t , $t_0 < t < T$.**
- At time t : Close out a FS by adding a FP for the same T .
- At time T :
 - Payoff from FP: $S_T - F_{t,T}$
You will buy one unit of the underlying and pay $F_{t,T}$ for it.
 - Payoff from FS: $F_{t_0,T} - S_T$
You will sell one unit of the underlying and receive $F_{t_0,T}$ for it.
 - Sum of the payoffs: $S_T - F_{t,T} + F_{t_0,T} - S_T = F_{t_0,T} - F_{t,T}$
- Discounting with the risk-free rate gives the present value for a FS:

$$PV_t = (F_{t_0,T} - F_{t,T})e^{-rT}$$

- Note: $PV_{t_0} = (F_{t_0,T} - F_{t_0,T})e^{-rT} = 0$

134
$$PV_T = F_{t_0,T} - F_{T,T} = F_{t_0,T} - S_T = CF_T$$



Arbitrage free forward price

- **Underlying = non-dividend paying security**

- $F_{t_0,T} = S_{t_0}e^{rT}$... **no arbitrage condition**

- Suppose $F_{t_0,T} < S_{t_0}e^{rT}$ ↻ **arbitrage strategy**

- $t=t_0$

- Long forward: 0
- Sell security short: $+S_{t_0}$
- Invest at r : $-S_{t_0}$
- Sum: 0

- $t=T$

- Fulfill forward: $-F_{t_0,T}$
- Receive from investment: $+S_{t_0}e^{rT}$
- Sum = Arbitrage profit: $S_{t_0}e^{rT} - F_{t_0,T}$

Arbitrage free forward price

- Suppose $F_{t_0,T} > S_{t_0}e^{rT}$ ↻ **arbitrage strategy**

- $t=t_0$

- Short forward: 0
- Buy security: $-S_{t_0}$
- Credit at r : $+S_{t_0}$
- Sum: 0

- $t=T$

- Fulfill forward: $+F_{t_0,T}$
- Credit repayment: $-S_{t_0}e^{rT}$
- Sum = Arbitrage profit: $F_{t_0,T} - S_{t_0}e^{rT}$



Arbitrage free forward price

- **Underlying = dividend-paying security**
- $F_{t_0,T} = (S_{t_0} - Z)e^{rT}$... **no arbitrage condition**
- Z ... present value of the dividends during T
- Suppose $F_{t_0,T} < (S_{t_0} - Z)e^{rT}$ ↻ **arbitrage strategy**
 - $t=t_0$
 - Long forward: 0
 - Sell security short: $+S_{t_0}$
 - Invest Z at r : $-Z$
 - Invest $S_{t_0} - Z$ at r : $-(S_{t_0} - Z)$
 - Sum: 0
 - $t=T$
 - Fulfill forward: $-F_{t_0,T}$
 - Receive from investment: $+(S_{t_0} - Z)e^{rT}$
 - Sum = Arbitrage profit: $(S_{t_0} - Z)e^{rT} - F_{t_0,T}$



Arbitrage free forward price

- Suppose $F_{t_0,T} > (S_{t_0} - Z)e^{rT}$ \Downarrow **arbitrage strategy**

- $t=t_0$

- Short forward: 0
- Buy security: $-S_{t_0}$
- Credit at r : $+S_{t_0}$
- Sum: 0

- $t=T$

- Fulfill forward: $+F_{t_0,T}$
- Credit repayment: $-S_{t_0}e^{rT}$
- Dividends received and invested at r : $+Ze^{rT}$
- Sum = Arbitrage profit: $F_{t_0,T} - (S_{t_0} - Z)e^{rT}$

Arbitrage free forward price

- Underlying = commodity that is an investment asset, e.g. gold

- $F_{t_0,T} = (S_{t_0} + L)e^{rT}$... no arbitrage condition

L ... present value of the storage costs incurred during T

- Suppose $F_{t_0,T} < (S_{t_0} + L)e^{rT}$ ↻ **arbitrage strategy**

- $t=t_0$

- Long forward: 0
- Sell security short: $+S_{t_0}$
- Invest $S_{t_0} + L$ at r : $-(S_{t_0} + L)$
- Sum: 0

- $t=T$

- Fulfill forward: $-F_{t_0,T}$
- Receive from investment: $+(S_{t_0} + L)e^{rT}$
- Sum = Arbitrage profit: $(S_{t_0} + L)e^{rT} - F_{t_0,T}$



Arbitrage free forward price

- Suppose $F_{t_0,T} > (S_{t_0} + L)e^{rT}$ \Downarrow **arbitrage strategy**

- $t=t_0$

- Short forward: 0
- Buy security: $-S_{t_0}$
- Pay storage costs: $-L$
- Credit at r : $+S_{t_0}+L$
- Sum: 0

- $t=T$

- Fulfill forward: $+F_{t_0,T}$
- Credit repayment: $-(S_{t_0}+L)e^{rT}$
- Sum = Arbitrage profit: $F_{t_0,T} - (S_{t_0}+L)e^{rT}$



Arbitrage free forward price

- **Underlying = commodity that is an consumption asset, e.g. oil**
- $F_{t_0,T} \leq (S_{t_0} + L)e^{rT}$... **no arbitrage condition**

L ... present value of the storage costs incurred during T

\leq ... Individuals who keep consumption assets will probably do so because of its consumption value - not because of its value as an investment! Therefore, we are unable to exploit the arbitrage opportunity given $F_{t_0,T} < (S_{t_0} + L)e^{rT}$ which would require that we sell the underlying!



Arbitrage free forward prices

- General principle: $F_{t_0,T} = S_{t_0} e^{cT}$... **no arbitrage condition**

c ... cost of carry

- Non-dividend paying security:

$$c=r$$

- Dividend paying security:

$$c=r-(1/T)\ln(S_{t_0}/(S_{t_0}-Z))$$

- Dividend paying security, where the dividend is expressed as a proportion q of the spot price:

$$c=r-q$$

- Commodity (investment asset):

$$c=r-(1/T)\ln(S_{t_0}/(S_{t_0}+L))$$

- Commodity (investment asset), where the storage costs are expressed as a proportion u of the spot price:

$$c=u+r$$

- Currency, where the income r^* is the foreign currency risk-free interest rate:

$$c=r-r^*$$

Arbitrage free forward price

- **Underlying = exchange rate**

- $F_{t_0,T} = S_{t_0} e^{(r-r^*)T}$... **no arbitrage condition**

- **Spot price/rate, S_t**

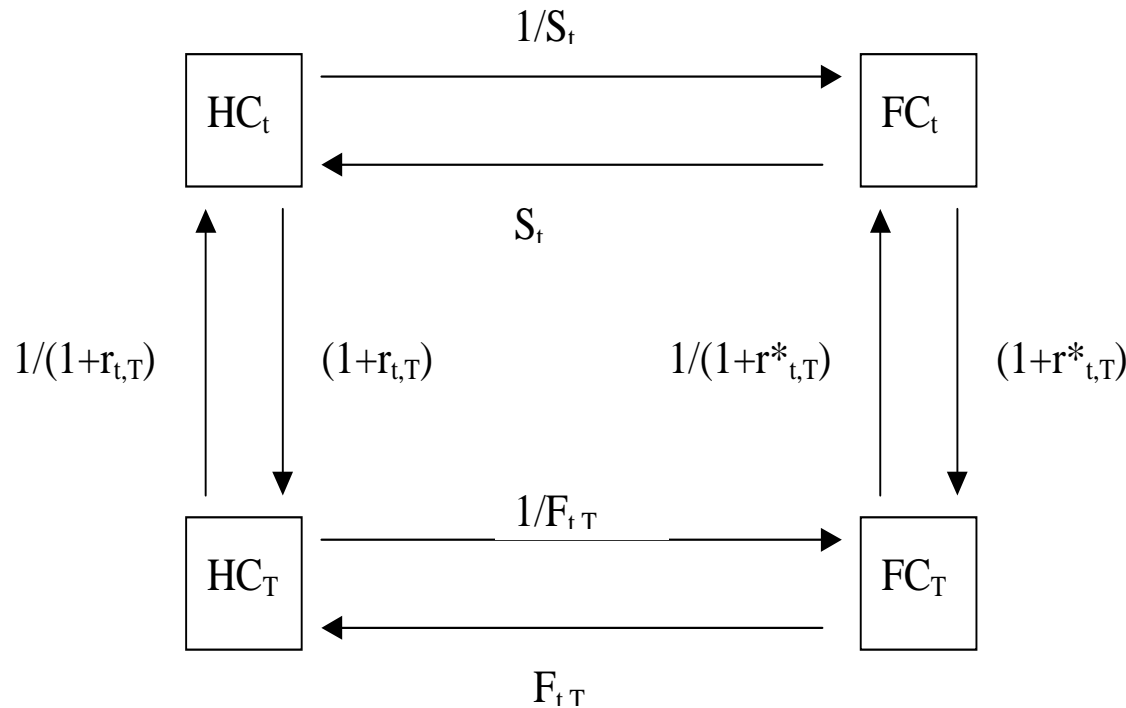
- **direct / right quote: #HC / 1FC**

- ... price of one unit of FC in units of HC
- ... buying or selling always refers to the currency in the denominator
- EUR / ice-cream = 2 ☹ price of 1 ice-cream is 2 EUR

- indirect / inverse quote: FC / HC

Arbitrage free forward price

- What does the FX-forward rate depend upon?



- No arbitrage condition: INTEREST RATE PARITY**

$$\left. \begin{aligned}
 1 &\geq e^{rT} * \frac{1}{F_{t,T}} * e^{-r^*T} * S_t \Rightarrow F_{t,T} \leq S_t e^{(r-r^*)T} \\
 1 &\geq \frac{1}{S_t} * e^{r^*T} * F_{t,T} * e^{-rT} \Rightarrow F_{t,T} \geq S_t e^{(r-r^*)T}
 \end{aligned} \right\} \Rightarrow F_{t,T} = S_t e^{(r-r^*)T}$$



Hedging examples

- Hedging a future FC inflow

- Hedging a future FC outflow

Problems

- Main problems associated with forwards (due to OTC)
 - non-standardized products; tailor made contracts
 - relative high transaction cost
 - relative high credit risk/default risk
 - ↳ illiquidity

- Would also be nice to have
 - relative low transaction cost
 - relative low credit risk/default risk
 - liquid secondary market
 - ↳ futures

These attributes of futures are achieved through standardization since standardization enables securities to be traded on an exchange

BUT

standardization also involves some difficulties especially w.r.t. hedging.

Problems

- Banks try to minimize the credit risk/default risk by
 - only dealing with well-known banks or corporations which have excellent reputation, “the club”
 - discouraging speculative positions
 - credit limits or margin requirements
 - issuing short-term contracts only (firms which want to hedge long-term positions must roll-over short-term contracts, but a rolled-over forward contract is an imperfect substitute for a long-term contract)

Overview

- Introduction
 - Definition
 - Clearing corporation
 - Marking to market

- Pricing

- Hedging

- Forwards vs. futures



Definition

- **Futures** are similar to forwards except the following features
 - **standardized** products (what?, where?, when?, how?)
 - typically involve **daily settlement** (“**marking to market**”)
 - **final settlement by offset** (i.e. no physical delivery)

Clearing corporation

- **Clearing corporation (CC)**

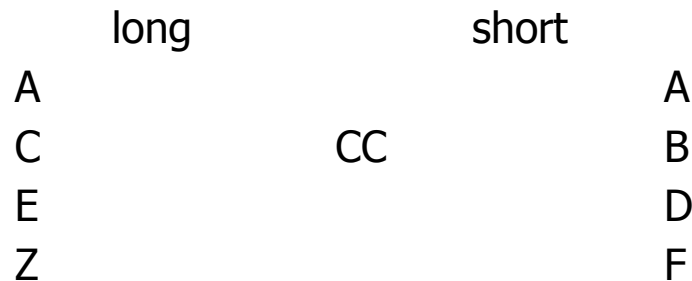
- Futures are not initiated between individuals or corporations BUT each party has contract with a CC, i.e.



... open interest = 3 contracts ... # of outstanding contracts

Note: if A defaults B isn't concerned; the only credit risk / default risk A faces is that the CC defaults ☹ virtually no credit risk

- The CC effectively clears, i.e. if A buys from B (like above) and then some time later sells to Z the CC cancels out both of A's contracts, i.e.



... open interest = 3 contracts ... # of outstanding contracts

- CC levies a small tax on all transactions ... insurance against default

Marking to market

- **Daily settlement (“marking to market”)** ... reduce the credit risk / default risk for the CC
 - Marking to market ... daily payment of the undiscounted change in the futures price; CFs arising from marking to market
 - long position: $CF_t = f_{t,T} - f_{t-1,T}$
total gain/loss = $f_{T,T} - f_{t_0,T} = S_T - f_{t_0,T}$
 - short position: $CF_t = f_{t-1,T} - f_{t,T}$
total CF = $f_{t_0,T} - f_{T,T} = f_{t_0,T} - S_T$
 - f_t ... settlement price: generally closing price or average price around the closing price
- Gain from defaulting for an investor = avoidance of a one-day marking-to-market outflow

Marking to market

- **Margin Account** ... to avoid the cost and inconvenience of frequent but small payments from marking to market, losses are allowed to accumulate to certain levels
 - **initial margin**
 - **maintenance margin** ... minimum amount of margin required
 - $\text{margin}_t = \text{margin}_{t-1} + \text{CF}_t$
 - if $\text{margin}_t < \text{maintenance margin}$ ↪ **margin call**
 - initial margin must be restored, i.e. $\text{initial margin} - \text{margin}_t =$ **"variation margin_t"**
 - if ignored: $\text{gain}_{\text{until } t} / \text{loss}_{\text{until } t}$ and margin_t
 - if $\text{margin}_t > \text{initial margin}$ ↪ withdrawing possible s.t. $\text{margin}_t \geq \text{initial margin}$



Example

- On **Wednesday**, an investor goes **long** a future on **HKD 1 million** at an **exercise price of TWD/HKD 3.3764**. The contract **expires on Tuesday next week**.
 - daily settlement prices TWK/HKD
Wed: 3.3776; Thu: 3.3421; Fri: 3.3990; Mon: 3.3428; Tue: 3.3290
 - initial margin = TWD 200000
 - maintenance margin = TWD 150000
 - The investor withdraws money from the margin account whenever she is entitled to do so
- At the close of each day, show the position of the margin account, the size of the margin call if required, and withdrawals of the investor. Also calculate the final settlement.



Future prices vs. forward prices

- Marking to market creates **interest rate risk**
 - $F=f$ if $r = \text{constant}$ / deterministic
 - $f < F$ if $\rho(\text{underlying price}, r) < 0$
 - if underlying inc. \Downarrow r dec. and gain; gains are invested at low rates
 - if underlying dec. \Downarrow r inc. and loss; losses have to be financed at high rates
 - $f > F$ if $\rho(\text{underlying price}, r) > 0$
 - if underlying inc. \Downarrow r inc. and gain; gains are invested at high rates
 - if underlying dec. \Downarrow r dec. and loss; losses can be financed at low rates
- In practice, differences are very small \Downarrow **forward and futures prices are usually assumed to be the same!**

Hedging

- ... some problems due to standardization
 - maturity mismatch ... “delta hedge”
 - underlying mismatch ... “cross hedge”
 - maturity mismatch and underlying mismatch ... “delta-cross hedge”
 - contract size mismatch
- 🔄 some risk will remain ... **“basis risk”**

Forwards vs. futures

Futures

- standardized products; exchange-traded
- relative low transaction cost
- relative low credit risk/default risk
- ⬇ liquid
 - ruin risk: cash flow problems may result from marking to market
 - interest rate risk due marking-to-market
 - limited choice of contracts
 - usually short maturities
 - usually no (physical) delivery just cash settlement (often prior to maturity)
- better for speculating

Forwards

- non-standardized products; tailor made contracts; OTC
- relative high transaction cost
- relative high credit risk/default risk
- ⬇ illiquid
 - not applicable
 - not applicable
 - tailor made contracts
 - longer maturities possible
 - usually delivery or cash settlement
- better for hedging

Overview

- Introduction
- Some profit/loss diagrams
- Pricing
 - No-arbitrage conditions beside the PCP
 - Factors affecting option prices
 - Discrete time: Binomial asset pricing model
 - Continuous time: Black-Scholes model
- Some exotic options
- Option markets

What is an option?

- The purchase of an options contract gives the buyer the **right to buy (call options contract) or sell (put options contract)** some other asset at a prespecified time and a prespecified price.
- The **underlying asset** can be any asset with a well-defined value or price. Examples are options on individual stocks, indices, futures contracts, bonds, currencies, other options, etc.



Difference betw. options and futures (forwards)

- An options contract **does not represent an obligation** to buy or sell the underlying asset, unlike the case of futures (forwards) contracts.
- As a result of that, the **value of an option can be positive, or at worst, zero.**

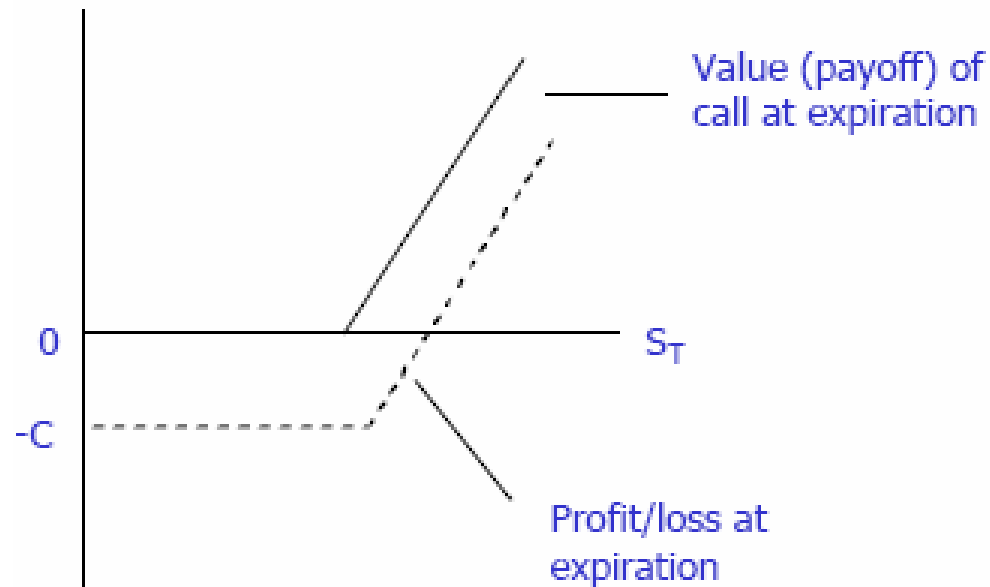


“American” vs. “European” options

- **“European” style options** provide the right to exercise only at the expiration date.
- **“American” options** give the right to buy or sell the underlying asset at any time on or before a prespecified future date (called the expiration date); **“early exercise”**.

Long call option

- Payoff and profit/loss diagram



- Profit / loss table

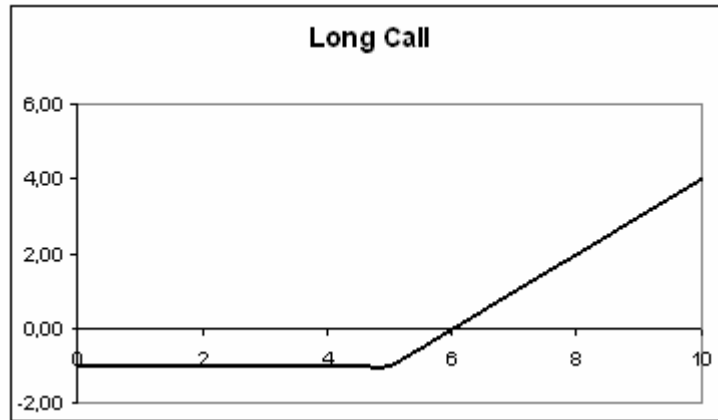
	<u>If $S_T = E$</u>	<u>If $S_T > E$</u>
Profit/loss	$-C$	$(S_T - E) - C$

- Algebraic representation

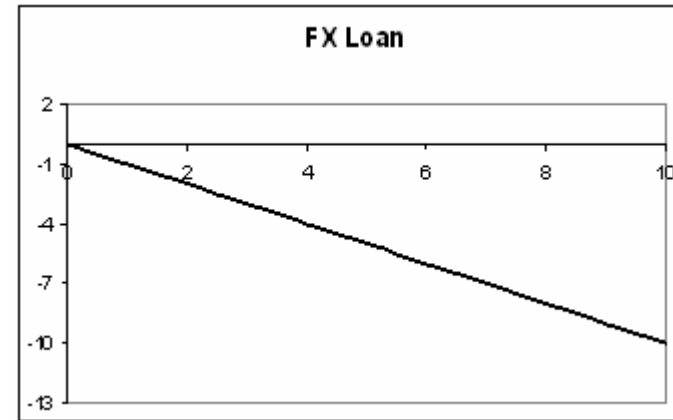
$$\text{Profit/loss} = \max \{0, S_T - E\} - C$$

Example: Hedging a FX-Loan

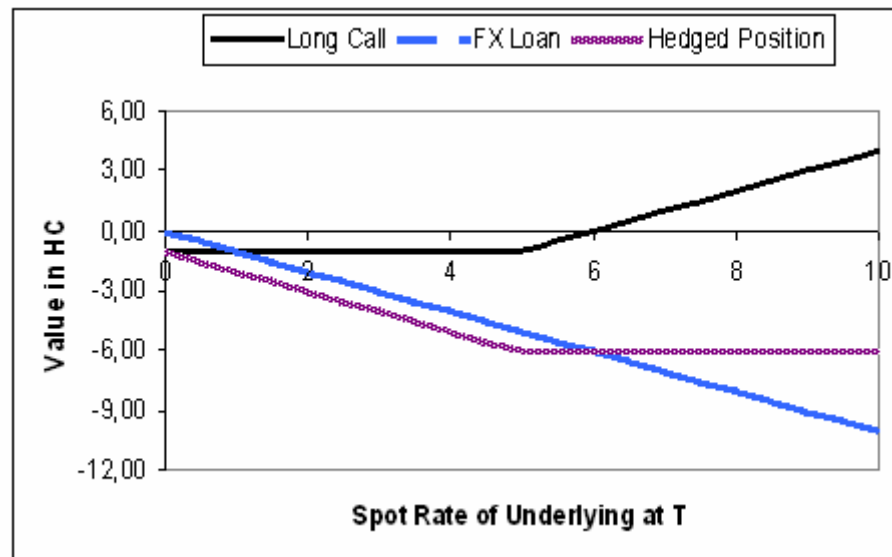
- Profit/loss diagram



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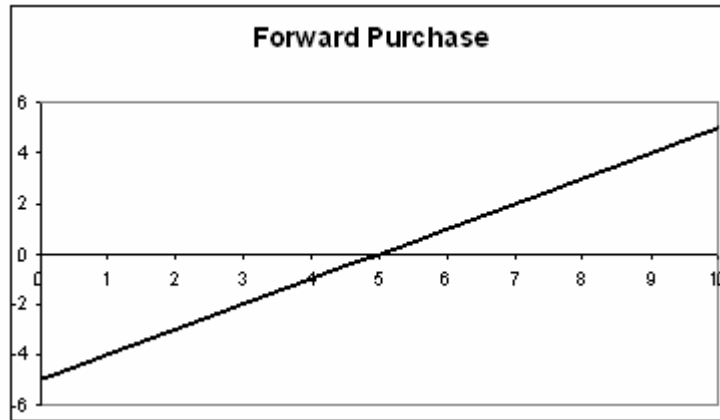


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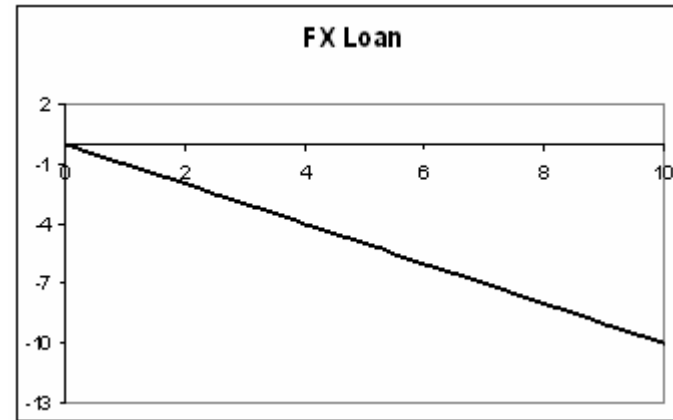


Example: Hedging a FX-Loan

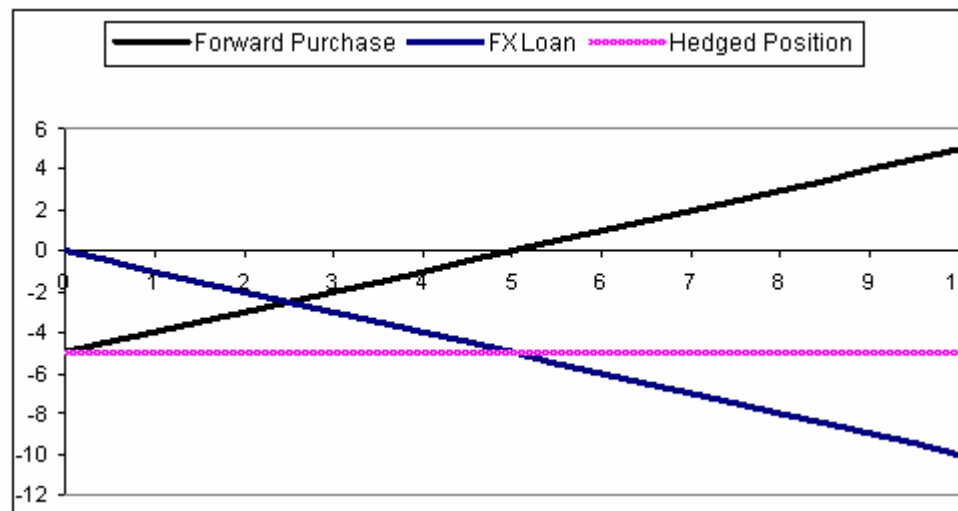
▪ Profit/loss diagram



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Remarks

- These simple profit/loss representations do not take into account:
 - The time value of money. In other words, the fact that the cost of the call is paid prior to the payoff at expiration.
 - Taxes.
 - Transaction costs.
- **In/at/out of the money**
 - At the money: Current price of underlying equals exercise price.
 - In the money: Immediate exercise would result in a profit.
 - Out of the money: Immediate exercise would result in a loss.
- **Intrinsic value of an option** = profit / loss if exercised immediately
- **Time value**: There is always a positive probability of a favorable underlying price movement.
- **Option value = Intrinsic value of an option + time value**

Short call option

- Payoff and profit/loss diagram
- Profit / loss table
- Algebraic representation

Short call option

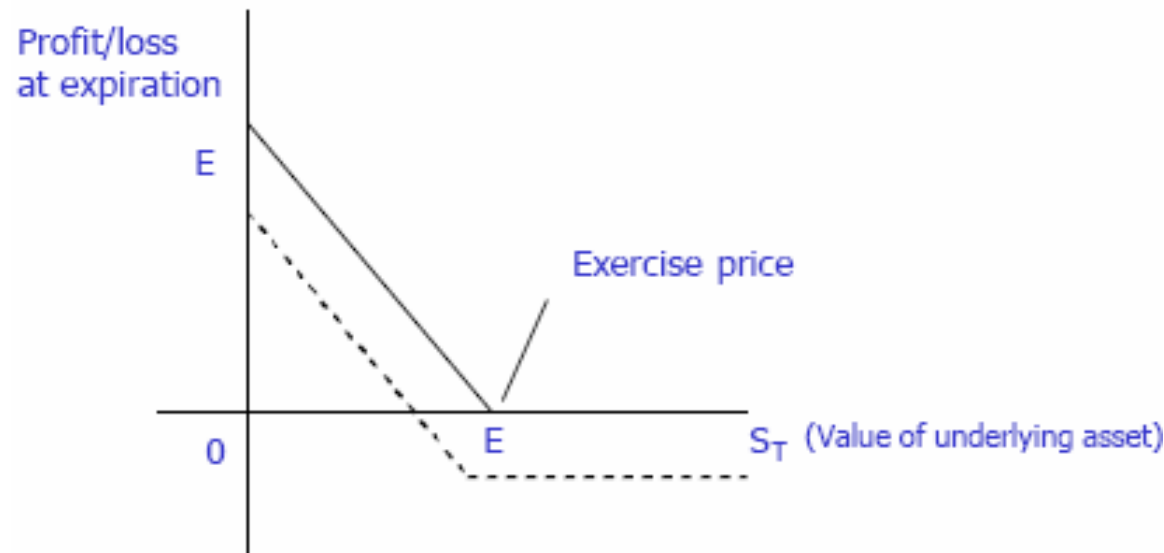
- The payoff and profit/loss diagram for the writer is exactly the opposite to that of the buyer.
- **Writing a call is a contingent obligation for which the writer is compensated by the sale price of the call.**

Remarks

- The payoffs to the writer and buyer of an option are perfectly negatively correlated; i.e., the options-related wealth positions of the buyer and writer always sum to zero.
- For this reason, options are not included in the market portfolio “M”, as they are in zero net supply.

Long put option

- Payoff and profit/loss diagram



- Profit / loss table

	<u>If $S_T = E$</u>	<u>$S_T > E$</u>
Profit/loss to put ownership	$(E - S_T) - P$	$-P$

- Algebraic representation

168 Profit/loss = $\max \{0, E - S_T\} - P$

Short put option

- Payoff and profit/loss diagram
- Profit / loss table
- Algebraic representation

Short put option

- The payoff or profit/loss diagram for the writer is exactly the opposite to that of the buyer.
- **Writing a put is a contingent obligation for which the writer is compensated by the sale price of the put.**

Portfolio insurance

- The return relationship of a put and the underlying asset are fundamentally different. This makes puts ideal instruments for insuring against price declines.

- Example

- 1 long share of stock, 1 long put on the stock with exercise price E
- Payoff table of a hedge

	$S_T = E$	$S_T > E$
Stock	S_T	S_T
Put	$E - S_T$	0
	E	S_T

- The portfolio's value is thus bounded below by the exercise price E .

Portfolio insurance

- Thus, portfolio insurance is a strategy that offers “insurance policy” on an asset. It works similarly to the previous example.
- Portfolio insurance offers a “**floor**” on the value of the portfolio.

Example: Portfolio insurance

- Suppose a portfolio has a value of \$100.
- Suppose a put option with exercise price (strike price) of \$100 has a sensitivity (delta) to changes in the value of the portfolio of -0.6. In other words, the option's value swings \$0.60 for every dollar change in portfolio value, but in an opposite direction.
- Suppose the stock price falls by 2%.
- What is the profit/loss on the portfolio that includes the put?

Example: Portfolio insurance

- Profit / loss table

Loss on stocks:	2% of \$100	= \$2.00
+Gain on put:	0.6 x \$2.00	= \$1.20
Net loss:		<hr/> = \$0.80

- In other words, including the put in the portfolio limits the loss to \$0.80 for every \$2.00 lost on the “uninsured” portfolio.



Example: Synthetic portfolio insurance

- We know that the delta of the put option is -0.6
- Buy delta shares, i.e. sell 60% of the shares value
- Put proceeds in T-bills
- Profit / loss table

Loss on stocks:	2% of \$40	= \$0.80
Loss on bills:		= <u>0</u>
Net loss:		= \$0.80

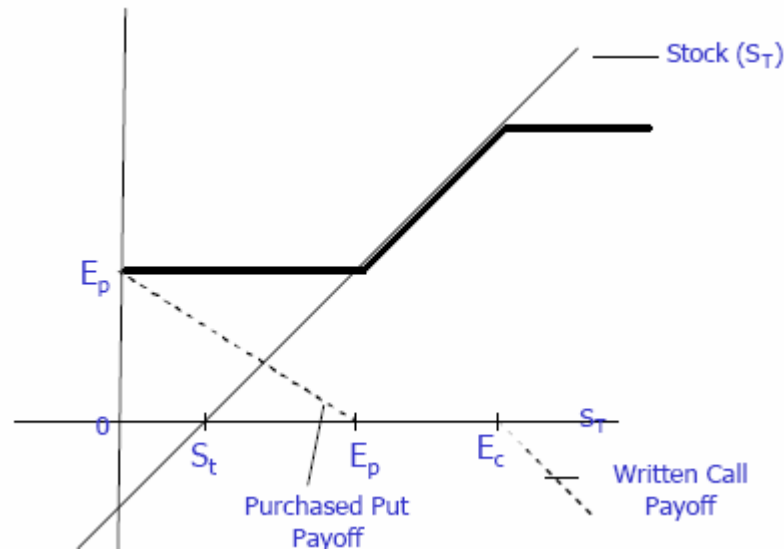


Portfolio insurance: The “collar”

- Portfolio insurance is used to ensure against declines in asset values. Sometimes, we can reduce the cost of such insurance by simultaneously writing calls.
- This is the idea of the “collar”.
- A **collar** is a portfolio of
 - The **underlying asset**
 - A **written call on the asset** with exercise price E_C
 - A **purchased put on the asset** with exercise price E_P
- The idea is to sell off some of the upward potential (in the form of the written call) in order to reduce the insurance costs (the price of the put), i.e. $E_C > E_P$.

Portfolio insurance: The "collar"

▪ Payoff diagram



▪ Payoff table

	$S_T = E_p$	$E_p < S_T < E_c$	$S_T = E_c$
Stock	S_T	S_T	S_T
Put	$E_p - S_T$	0	0
Call	0	0	$-(S_T - E_c)$
	E_p	S_T	E_c

Put-call parity

- Consider the following portfolio:
 - Buy 1 share of the stock
 - Write one call on the stock with exercise price E
 - Buy one put on the stock with the same exercise price E
 - Borrow $PV(E)$, where E is the common exercise price of the put and the call.

Put-call parity

- Price and payoff table

	Cost Today	$S_T = E$	$S_T > E$
Write a call	$+ C_t$	0	$E - S_T$
Buy a put	$- P_t$	$E - S_T$	0
Buy one share	$- S_t$	S_T	S_T
		<hr/>	<hr/>
		E	E
Get a loan	$+PV(E)$	-E	-E
		<hr/>	<hr/>
		0	0

- Note
 - No matter what happens at expiration, this portfolio pays E.
 - In the absence of arbitrage opportunities, it must sell today for a price equal to $PV(E)$.

Put-call parity

$$-C_t + P_t + S_t = PV_t(E) \text{ or}$$

$$C_t = P_t + S_t - PV_t(E) \text{ or}$$

$$P_t = C_t - S_t + PV_t(E)$$

Note:

$$P_T - C_T = PV_T(E) - S_T$$

$$P_T - C_T = E - S_T$$

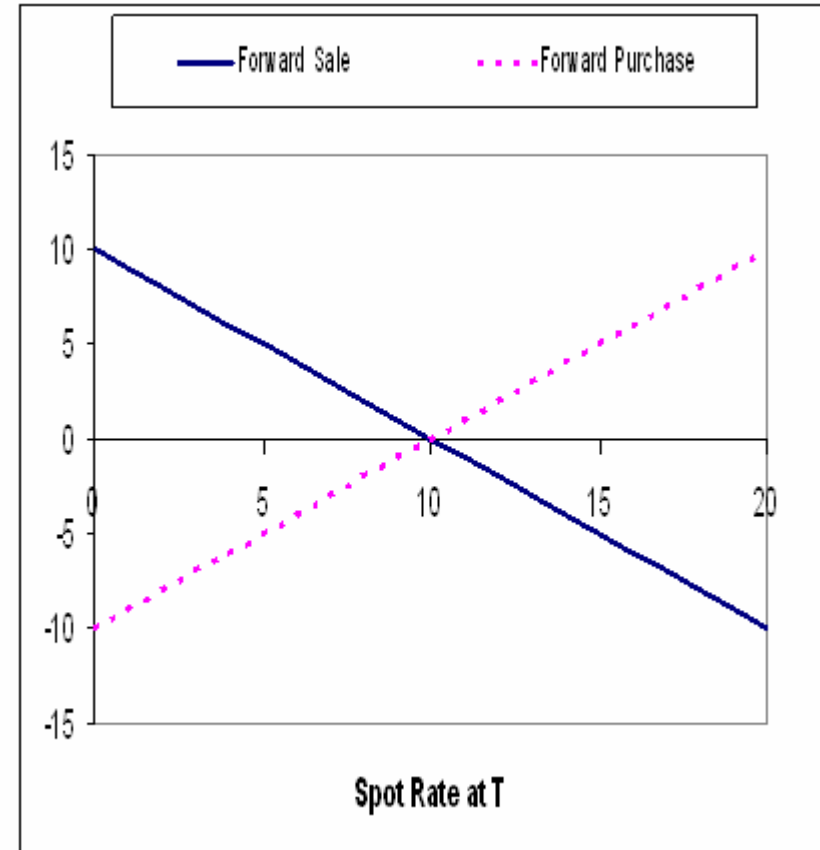
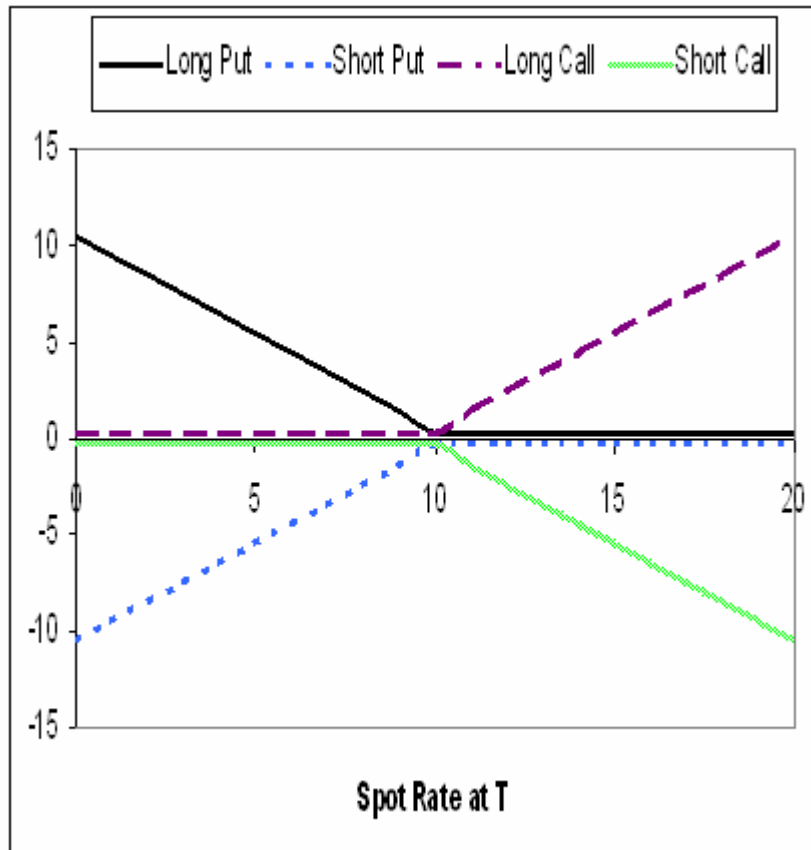
$$= F_{t0,T} - S_T$$

= CF_T of a FS and therefore by the LAW OF ONE PRICE:

$$P_t - C_t = PV_t \text{ of a FS, which is given by } PV_t = (F_{t0,T} - F_{t,T})e^{-rT}$$

Put-call parity

- Payoff diagram



Bull spread

- with calls: long 1 call, $E=E_1$ and short 1 call, $E=E_2$, where $E_2 > E_1$
- Profit/loss diagram

Bull spread

- with puts: long 1 put, $E=E_1$ and short 1 put, $E=E_2$, where $E_2 > E_1$
- Profit/loss diagram

Bear spread

- with calls: short 1 call, $E=E_1$ and long 1 call, $E=E_2$, where $E_2 > E_1$
- Profit/loss diagram

Bear spread

- with puts: short 1 put, $E=E_1$ and long 1 put, $E=E_2$, where $E_2 > E_1$
- Profit/loss diagram

Butterfly spread

- with calls: long 1 call, $E=E_1$; short 2 calls, $E=E_2$ and long 1 call, $E=E_3$, where $E_3 > E_2 = 0.5(E_1 + E_3) > E_1$
- Profit/loss diagram

Butterfly spread

- with puts: long 1 put, $E=E_1$; short 2 puts, $E=E_2$ and long 1 put, $E=E_3$, where $E_3 > E_2 = 0.5(E_1 + E_3) > E_1$
- Profit/loss diagram

Straddle

- long: long 1 call, E and long 1 put, E
- Profit/loss diagram

Straddle

- short: short 1 call, E and short 1 put, E
- Profit/loss diagram



Strangle

- long: long 1 put, E_1 and long 1 call, E_2 , where $E_2 > E_1$
- Profit/loss diagram

Strangle

- short: short 1 put, E_1 and short 1 call, E_2 , where $E_2 > E_1$
- Profit/loss diagram

Outline

- No-arbitrage conditions beside the PCP
- Factors affecting option prices
- Discrete time: Binomial asset pricing model
- Continuous time: Black-Scholes model

European call

- $\max(0, S_t - Ee^{-rT}) \leq c \leq S_t \dots$ NAC
- Suppose $0 < c < S_t - Ee^{-rT} \Rightarrow$ arbitrage strategy

European call

- Suppose $c > S_t > 0 \Rightarrow$ arbitrage strategy

European put

- $\max(0, Ee^{rT}-S_t) \leq p \leq Ee^{rT}$... NAC
- Suppose $0 < p < Ee^{rT}-S_t$ \Downarrow arbitrage strategy

European put

- Suppose $p > Ee^{rT} > 0 \Rightarrow$ arbitrage strategy

Factors affecting option prices

<i>Variable</i>	<i>European Call</i>	<i>European Put</i>	<i>American Call</i>	<i>American Put</i>
Stock price	+	-	+	-
Strike price	-	+	-	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	-	+	-
Dividends	-	+	-	+

One-period binomial model

- Time 0: S_0 ... price per share, a positive quantity known at time zero.
- Time 1: The price per share will be one of two positive values: $S_1(H)$ or $S_1(T)$.
- Assume:
 - Probability of head (stock price increase), p , is positive.
 - Probability of tail (stock price decrease), $q = (1-p)$, is also positive.
- The outcome of the coin toss, and hence $S_1(H)$ or $S_1(T)$, is known at time one but not at time zero, it is random.
- $u = S_1(H) / S_0 > 0$ and $d = S_1(T) / S_0 > 0$

One-period binomial model

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One-period binomial model

- **To rule out arbitrage we must assume: $0 < d < 1+r < u$.**
 - Positivity of stock prices $\Leftrightarrow d > 0$.
 - If $d \geq 1+r \Leftrightarrow$ Arbitrage strategy:
 - Time 0: Borrow from the money market in order to buy the stock.
 - Time 1: Even in the worst case, the value of the stock will be higher than or equal to the value of the money market debt and has a positive probability of being strictly higher since $u > d \geq 1+r$.
 - If $u \leq 1+r \Leftrightarrow$ Arbitrage strategy:
 - Time 0: Sell the stock short and invest the proceeds in the money market.
 - Time 1: Even in the best case, the cost canceling the short position will be less than or equal to the value of the money market investment and has a positive probability of being strictly less since $d < u \leq 1+r$.

The replicating portfolio

- Under the binomial model, **a derivative will assume at most two values** ☹ **we need at most two other securities to match its payoff exactly!**
- The arbitrage pricing theory approach to the derivative-pricing problem is to **replicate the derivative by trading in the stock and the money market.**
- **The initial wealth needed to set up the replicating portfolio is the no-arbitrage price of the derivative at time zero!**
 - Derivate price in the market $>$ no-arbitrage price ☹ Arbitrage strategy:
 - Time 0: Sell the derivative short, set up the replicating portfolio and invest the rest in the money market.
 - Time 1: Regardless of how the stock price evolved, we have a zero net position in the derivative and the money market investment.
 - Derivate price in the market $<$ no-arbitrage price ☹ Arbitrage strategy:
 - Time 0: Buy the derivative, set up the reverse of the replicating portfolio and invest the rest in the money market.
 - Time 1: Regardless of how the stock price evolved, we have a zero net position in the derivative and the money market investment.



The replicating portfolio

- Time 0: V_0 ... no-arbitrage price of the derivative, to be determined.
- Time 1: The payoff of the derivative will be one of two values: $V_1(H)$ or $V_1(T)$.
-

The replicating portfolio

- Consider a portfolio of stock (number of stocks = ϕ_0) and money market account (amount invested at the risk-free rate = B_0).
- If the portfolio is to replicate the payoff to the derivative, it must be that $\phi_0 S_1 + (1+r) B_0 = V_1$, i.p.
 - (i) $\phi_0 S_1(H) + (1+r) B_0 = V_1(H)$, i.e. $\phi_0 uS_0 + (1+r) B_0 = V_1(H)$ and
 - (ii) $\phi_0 S_1(T) + (1+r) B_0 = V_1(T)$, i.e. $\phi_0 dS_0 + (1+r) B_0 = V_1(T)$.
 ... Two equations, two unknowns (ϕ_0 and B_0).

Note: $B_0 = X_0 - \phi_0 S_0$ We begin with wealth X_0 and buy ϕ_0 shares of stock at time zero, leaving us with a cash position $B_0 = X_0 - \phi_0 S_0$. The value of our portfolio of stock and money market account at time one is given by the wealth equations (i) and (ii).

The replicating portfolio

- From the wealth equations (i) and (ii) we get:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{(u - d)S_0} \quad \& \quad B_0 = \frac{uV_1(T) - dV_1(H)}{(u - d)(1 + r)}.$$

- Recall: **The initial wealth, X_0 , needed to set up the replicating portfolio is the no-arbitrage price of the derivative at time zero. Thus,**

$$B_0 = X_0 - \Delta_0 S_0 \quad \Leftrightarrow \quad X_0 = \Delta_0 S_0 + B_0 = V_0, \text{ by no-arbitrage!}$$

The replicating portfolio

- Let us now consider an alternative way to solve the wealth equations (i) and (ii).
- We rewrite the wealth equation as follows:

$$\begin{aligned}V_1 &= \Delta_0 S_1 + (1+r)B = \\ &= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = \\ &= (1+r)X_0 + \Delta_0 (S_1 - (1+r)S_0) \\ \Rightarrow \frac{V_1}{1+r} &= X_0 + \Delta_0 \left(\frac{S_1}{1+r} - S_0 \right), \text{ i.p.}\end{aligned}$$

The replicating portfolio

$$(i') \quad \frac{V_1(H)}{1+r} = X_0 + \Delta_0 \left(\frac{S_1(H)}{1+r} - S_0 \right) \quad \&$$

$$(ii') \quad \frac{V_1(T)}{1+r} = X_0 + \Delta_0 \left(\frac{S_1(T)}{1+r} - S_0 \right).$$

... Two equations, two unknowns (Δ_0 and X_0).

- Multiply (i') by a number p' , (ii') by a number $q' = 1-p'$ and add them to get

$$\frac{\tilde{p}V_1(H) + \tilde{q}V_1(T)}{1+r} = X_0 + \Delta_0 \left(\frac{\tilde{p}S_1(H) + \tilde{q}S_1(T)}{1+r} - S_0 \right).$$

The replicating portfolio

- If we choose p' so that

$$S_0 = \frac{\tilde{p}S_1(H) + \tilde{q}S_1(T)}{1+r} \quad (*)$$

$$\Rightarrow X_0 = \frac{\tilde{p}V_1(H) + \tilde{q}V_1(T)}{1+r} = V_0, \quad (**)$$

where the last equality comes from the no-arbitrage argument.

- We can solve for p' and q' directly from the equation (*) in the form

$$S_0 = \frac{\tilde{p}S_1(H) + \tilde{q}S_1(T)}{1+r} = \frac{\tilde{p}uS_0 + \tilde{q}dS_0}{1+r} = \frac{S_0((u-d)\tilde{p} + d)}{1+r}$$

$$\Rightarrow \tilde{p} = \frac{1+r-d}{u-d} \quad \& \quad \tilde{q} = 1 - \tilde{p}.$$

The replicating portfolio

- The replication argument depends on several assumptions:
 - **Short positions are allowed** (unlimited credit).
 - **Shares of stock can be subdivided** for sale or purchase.
Essentially satisfied because option pricing and hedging (replication) typically involve lots of options.
 - **The interest rate for investing is the same as the interest rate for borrowing**; we use a constant risk-free rate which is assumed to be the same for all maturities.
Is close to being true for large institutions.
 - **The purchase price of stock is the same as the selling price**, i.e. there is zero bid-ask spread.
Is not satisfied in practice.
 - **No transaction costs, taxes, ...**
 - At any time, **the stock can take only two possible values in the next period.**
In the Black-Scholes model, this assumption is replaced by the assumption that the stock price is a geometric Brownian motion. Empirical studies of stock price returns have consistently shown this not to be the case!

Risk-neutral probabilities

- Properties of p' and q'
 - p' and q' are positive:
due to the no-arbitrage assumption, $0 < d < 1+r < u$.
 - p' and q' sum to one:
 $p' + q' = p' + 1 - p' = 1$.
 - $0 < p' < 1$:
 $d < 1+r < u$,
 $d < \tilde{p}(u-d) + d < u$,
 $d - d < \tilde{p}(u-d) < u - d$,
 $0 < \tilde{p} < 1$.

☞ We can regard p' and q' as probabilities of head and tail, respectively.

Risk-neutral probabilities

↳ We can rewrite (*) and (**) as

$$S_0 = \frac{\tilde{p}S_1(H) + \tilde{q}S_1(T)}{1+r} = \frac{\tilde{E}(S_1)}{1+r} \quad \&$$
$$V_0 = \frac{\tilde{p}V_1(H) + \tilde{q}V_1(T)}{1+r} = \frac{\tilde{E}(V_1)}{1+r},$$

where $\tilde{E}(\cdot)$ denotes the expected value under \tilde{p} and \tilde{q} .

Risk-neutral probabilities

- Note that p' and q' satisfy (*),

$$S_0 = \frac{\tilde{p}S_1(H) + \tilde{q}S_1(T)}{1+r} \quad \text{o.e.} \quad S_0(1+r) = \tilde{p}S_1(H) + \tilde{q}S_1(T), \text{ i.e.}$$

the average rate of growth of the stock under p' and q' is exactly the same as the rate of growth of the money market account.

- If this would be the case then investors must be neutral about risk - they do not require compensation for assuming it (risk averse), nor are they willing to pay for it (risk loving). This is simply not the case ☹ p' and q' are not the actual probabilities, which we call p and q , but rather so-called risk-neutral probabilities!

Note

- The valuation seems not to take into account the expected rate of return of the underlying asset!
 - This appears counterintuitive, but the probability of an up or down move is already incorporated in today's stock price.
 - It turns out, that the expected return needs not to be taken into account elsewhere.

The actual probabilities of up and down moves are irrelevant. What matters is the size of the two possible moves (the values u and d).

I.e. the prices of derivative securities depend on the set of possible stock price paths but not on how probable these paths are.



Example: (European or American) call

- Consider a traded asset (stock) with current price $S = 50\text{USD}$, a **call (European or American)** with $E = 55\text{USD}$, $T = 1\text{year}$, $r = 10\%\text{p.a.}$, $u = 1.3$ and $d = 0.77$. Determine the price of the derivative?



Example: (European or American) put

- Consider a traded asset (stock) with current price $S = 50\text{USD}$, a **put (European or American)** with $E = 55\text{USD}$, $T = 1\text{year}$, $r = 10\%\text{p.a.}$, $u = 1.3$ and $d = 0.77$. Determine the price of the derivative?

Example: Arbitrage possibility

- Consider the example on the previous page but suppose now that you observe the price of the put to be 6USD! What would you conclude?



Multi-period binomial asset pricing model

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Multi-period binomial asset pricing model

- After a coin toss, the agent can readjust her replicating portfolio. Thus, in order to determine the no-arbitrage price of the derivative at time zero we can proceed via **backward induction, i.e. we determine the no-arbitrage price of the derivative for each sub-tree starting at the very right and work “backward” to the very left.**
- Additional assumption as compared to the one-period case:
 - **u and d are constant:** Since u and d measure the volatility of the underlying, we implicitly assume that this volatility is constant!
This is empirically not justified!
 - After a coin toss, the agent can **readjust her replicating portfolio at no cost.**



Example: European put

- Consider a traded asset (stock) with current price $S = 100\text{USD}$, an **European put** with $E = 100\text{USD}$, $T = 1\text{year}$, 3 periods, $r = 5\%\text{p.a.}$, $S_1(H) = 125\text{USD}$ and $S_1(T) = 80\text{USD}$. Determine the price of the derivative?

American options

- Recall: **“American” options** give the right to buy or sell the underlying asset at any time on or before a prespecified future date (called the expiration date); **“early exercise”**.
- This implies that we need at least a 2-step binomial asset pricing model to value the possibility of early exercise.

Note: In a 1-step binomial asset pricing model American and European options will have the same value.

American options

- The procedure for valuing an American option is as follows:

- At every node also calculate the intrinsic value of the American option, $IV(.)$.

Recall: Intrinsic value of an option = profit / loss if exercised immediately.

- If $IV(.) > V(.)$ ⤴ Early exercise, i.e. we realize the intrinsic value. ⤴ Since everybody agrees with the fact that the $IV(.) > V(.)$ nobody will be willing to sell the option for less than its immediate exercise value therefore we continue our calculation with $IV(.)$ instead of $V(.)$.
- If $IV(.) < V(.)$ ⤴ Early exercise is not desirable. ⤴ We continue our calculation without changes!

Example: American put

- Consider a traded asset (stock) with current price $S = 100\text{USD}$, an **American put** with $E = 100\text{USD}$, $T = 1\text{year}$, 3 periods, $r = 5\%\text{p.a.}$, $S_1(H) = 125\text{USD}$ and $S_1(T) = 80\text{USD}$. Determine the price of the derivative?

Example: Lookback option

- Consider a traded asset (stock) with current price $S = 4\text{USD}$, $u = 2$, and $d = 0.5$, 3 periods and $r = 2.5\% \text{p.p.}$. The **Lookback option** pays off $V_3 = \max_{0 \leq n \leq 3} S_n - S_3$. Determine the price of the derivative?



Multi-period binomial asset pricing model

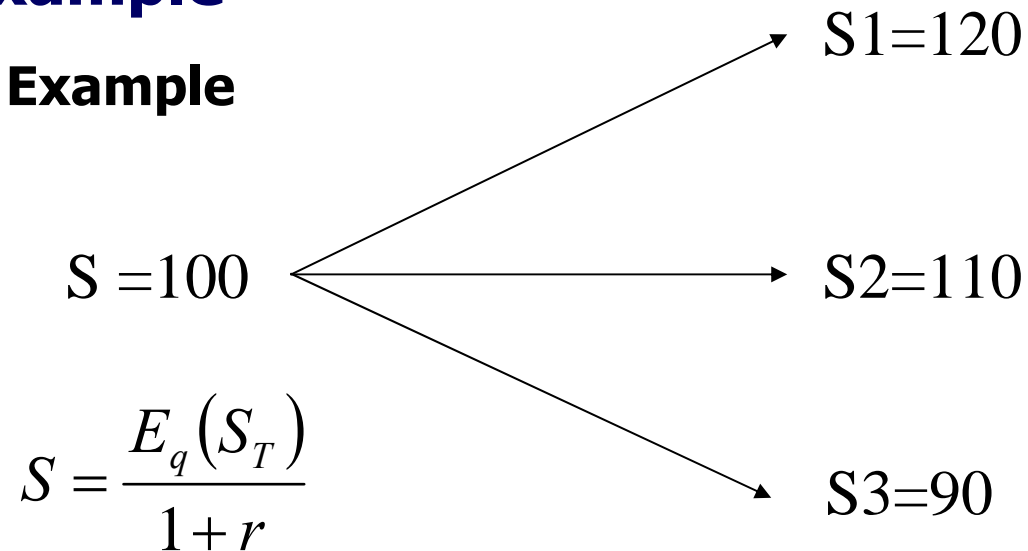
- If we choose $u = e^{\sigma\sqrt{\Delta t}}$, $d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$ and a continuous interest rate convention then as the number of periods goes to ∞ , the probability distribution for the value of the underlying asset approaches a normal distribution.
- The binomial model approximates the Black-Scholes model as $\Delta t \rightarrow 0$, the price of a call computed using the binomial model will approximate the Black-Scholes price.

Completeness

- The binomial asset pricing in this section is called **complete because every derivative can be replicated** by trading in the underlying stock and the money market.
- **In a complete market, every security has a unique price.**
- **Many markets are incomplete, and prices cannot be determined from no-arbitrage considerations alone. Utility based models are still the only theoretically defensible way of treating such markets.**

Example

- **Example**



$$S = \frac{E_q(S_T)}{1+r}$$

$$100 = \frac{\tilde{q}_1 \cdot 120 + \tilde{q}_2 \cdot 110 + (1 - \tilde{q}_1 - \tilde{q}_2) \cdot 90}{1.1}$$

$$\Rightarrow \tilde{q}_2 = 1 - \frac{3}{2} \cdot \tilde{q}_1$$

Take any q_1 such that the risk-neutral probabilities are between 0 and 1. Two possible choices for the risk-neutral measures would be (0,1,0) or (2/3,0,1/3).

The evolvement of r and S over time

- In the Black-Scholes option pricing model (1973), there are two securities, **a money market account which offers a constant risk-free interest rate and a stock** (just like in the binomial asset pricing model).

- The **money market account follows a deterministic process** such as:

$$\frac{dB_t}{B_t} = rdt$$

where r is the riskless interest rate, dt is a small time step, and dB_t is called the increment of B over the time interval $[t, t+dt]$.

- The **stock follows a geometric Brownian motion (GBM)** such as:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where μ is the constant mean of S , dt is a small time step, σ is the constant standard deviation of S , dS_t (dW_t) is called the increment of S (W) over the time interval $[t, t+dt]$, and **W is a Wiener process**. For any fixed time interval $[t, t+dt]$ the increment dS_t (dW_t) is a stochastic variable!

Wiener process

- Norbert Wiener, 1920: W_t is a **Wiener process**, i.p. W_t is a random (stochastic) real-valued continuous function (process) on $[0, \infty)$ such that:
 - $W_{t=0} = 0$,
 - $dW_t = W_{t+dt} - W_t \sim N(0, dt)$, and
 - *if the intervals $[t_1, t_2]$ and $[u_1, u_2]$ do not overlap, then the increments $dW_t = W_{t_2} - W_{t_1}$ and $dW_u = W_{u_2} - W_{u_1}$ are independent!*
- One realization of a Wiener process
- Some implied properties
 - **W is nowhere differentiable** due to its jaggedness which is a result of the independent increments
 - Since each increment of W is normal distributed **W itself is normal distributed**

Geometric Brownian motion

- $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$, where $dW_t \sim N(0, dt)$

$$E\left[\frac{dS_t}{S_t}\right] = E[\mu dt + \sigma dW_t] = E[\mu dt] + E[\sigma dW_t] = \mu dt + \sigma E[dW_t] = \mu dt + 0 = \mu dt$$

$$V\left[\frac{dS_t}{S_t}\right] = V[\mu dt + \sigma dW_t] = V[\mu dt] + V[\sigma dW_t] + 2 \text{cov}[\mu dt, \sigma dW_t] = 0 + \sigma^2 V[dW_t] + 0 = \sigma^2 dt$$

$$\Rightarrow \frac{dS_t}{S_t} \sim N(\mu dt, \sigma^2 dt) \ \& \ dS_t \sim N(\mu S_t dt, \sigma^2 S_t^2 dt)$$

- One realization of a geometric Brownian motion

Geometric Brownian motion

- $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$, where $dW_t \sim N(0, dt) \Rightarrow \frac{dS_t}{S_t} \sim N(\mu dt, \sigma^2 dt)$
- Note: The probability of what S does next depend only on the current state, t . This is called **Markov property**.

In other words: In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are.

- Over a small time interval $[t, t+dt]$ a GBM has the following **economic interpretation**:

stock return = mean return + volatility * normal random disturbance

- large volatility \Downarrow large random fluctuations
- small volatility \Downarrow small random fluctuations

History

- **Robert Brown, 1828:** Experimental study of the motion of particles suspended in a liquid (Albert Einstein, 1905 and Marian von Smoluchowski, 1906). The motion (stochastic process) became known as Brownian motion.
- **Louis Bachelier, *Théorie de la Speculation*, 1900:** S follows a Brownian motion such as:

$$dS_t = \mu dt + \sigma dW_t$$

- BUT a **Brownian motion** and herefore S_t may become negative.
- This difficulty is easily eliminated by assuming that the **logarithm of S_t** , rather than S_t itself, follows a **Brownian motion**. In this case we say that S_t follows a **geometric Brownian motion**.

$$d \ln S_t = \mu_0 dt + \sigma_0 dW_t \Leftrightarrow \frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Black-Scholes model: Main ideas

- The **BS analysis is analogous to the analysis used to value options in the binomial asset pricing model.**
- Consider a **portfolio consisting of a position in the option and a position in the underlying stock.**

The **derivative is defined in terms of the underlying** ⇨ **the derivative price should be highly correlated with the underlying price.**

In any short period of time, the price of a call option is perfectly positively correlated with the price of the underlying stock and the price of a put option is perfectly negatively correlated with the price of the underlying stock.

⇨ We should be able to **balance derivative against underlying in our portfolio, so as to cancel the randomness.** In other words, we should be able to choose the weights of the portfolio such that we get **a riskless portfolio.**

Black-Scholes model: Main ideas

Suppose, for example, that at a particular point in time we know that the delta of a call is given by 0.4 ($\Delta_c = dc/dS = 0.4$).

The riskless portfolio would consist of

- a long position in 0.4 share and
- a short position in 1 call option.

The gain or loss from the stock position always offsets the gain or loss from the option position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

I.e. Δ_c is the number of units of the underlying one should hold for each option shorted if one wants to obtain a riskless portfolio.

This is exactly the same reasoning as in the binomial asset pricing model! However, there is one important difference between the binomial asset pricing model and the BS model. **In the BS model, the position that is set up is riskless for only a very short period of time.**



Black-Scholes model: Main ideas

- **To remain riskless it must be frequently adjusted or “rebalanced”.** In particular, we would have to rebalance continuously!
- However, **we will obtain a riskless rate of return on our portfolio which by absence of arbitrage is equal to our riskless interest rate from the money market account.**
- **This is the key element in the BS arguments and leads to their pricing formulas.**



Black-Scholes model: Main ideas

- The riskless-portfolio argument depends on basically the same assumptions as the replicating portfolio argument in the binomial asset pricing model, except that we assume here that **the stock price follows a geometric Brownian motion**. (This implies continuous trading is assumed to be possible.)

Empirical studies of stock price returns have consistently shown this not to be the case!

Black-Scholes model: Pricing formula

- Black-Scholes formula

$$c_t = S_t N(d1) - E e^{-rT} N(d2)$$

$$p_t = E e^{-rT} N(-d2) - S_t N(-d1)$$

$$d1 = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d2 = d1 - \sigma\sqrt{T}$$

where $N(x)$ is cumulative standard normal distribution at x . In other words, it is the probability that a variable with a standard normal distribution, i.e. $N(0,1)$, will be less than x .

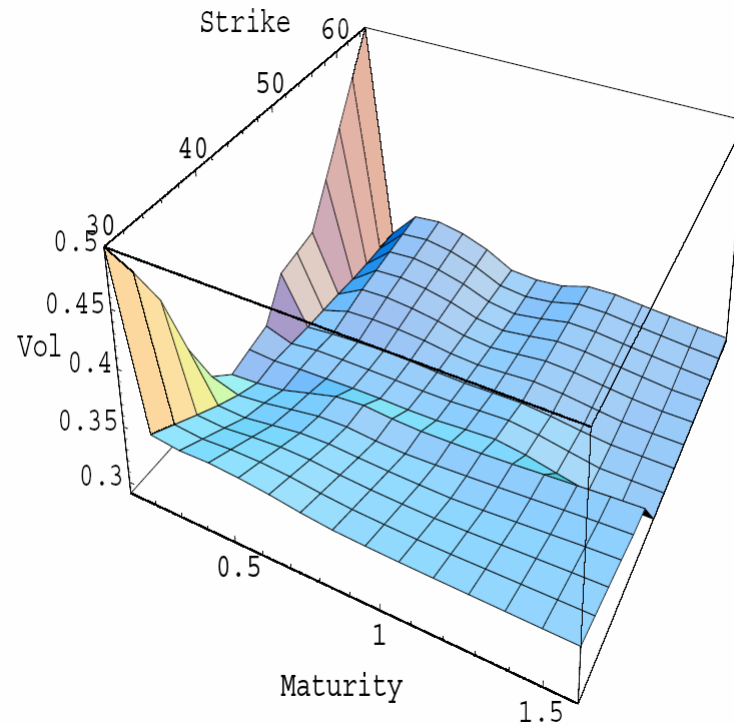
Note: The equations for the call price and the put price are of course related via the put-call parity, i.e. via $p_t = c_t - S_t + PV_t(E)$.

Black-Scholes model: Pricing formula

- **The volatility is the only parameter which can not be directly observed in the market** ↻ **We can either use**
 - **historical estimation OR**
 - **the “implied volatility”.**
- Implied volatility
 - One can use the BS formula to calculate the volatility given all other observed values (including the observed market price of the option). This volatility is then called the “implied volatility”.
 - Unfortunately, there exists no closed form solution for the implied volatility, i.e. we cannot rewrite the Black-Scholes pricing formula to get an expression for the implied volatility. However, one can use numerical procedures (e.g. Monte Carlo simulations) that provide solutions.

Black-Scholes model: Pricing formula

- The result is the following “volatility smile”:



Note that we assumed a constant volatility which is obviously an incorrect assumption.

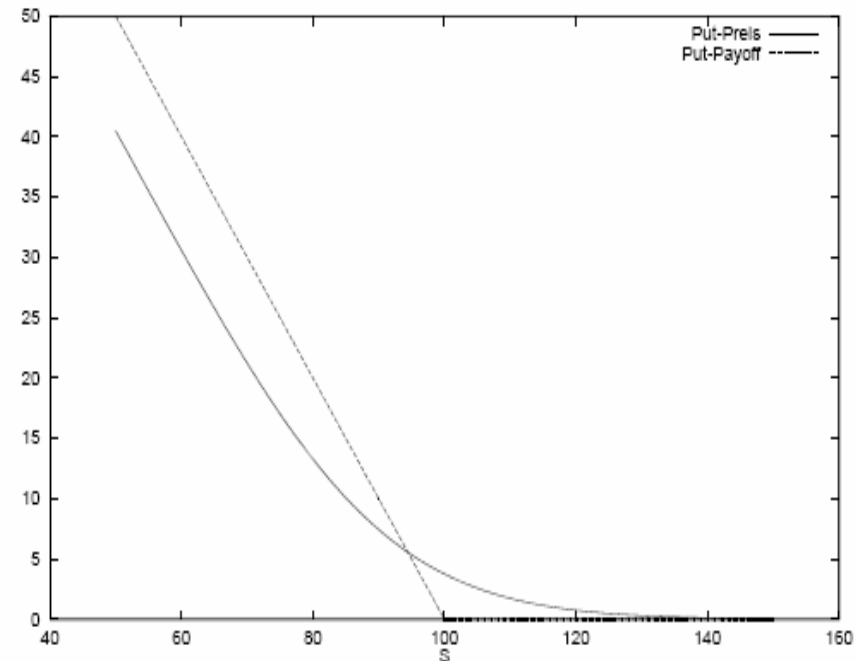
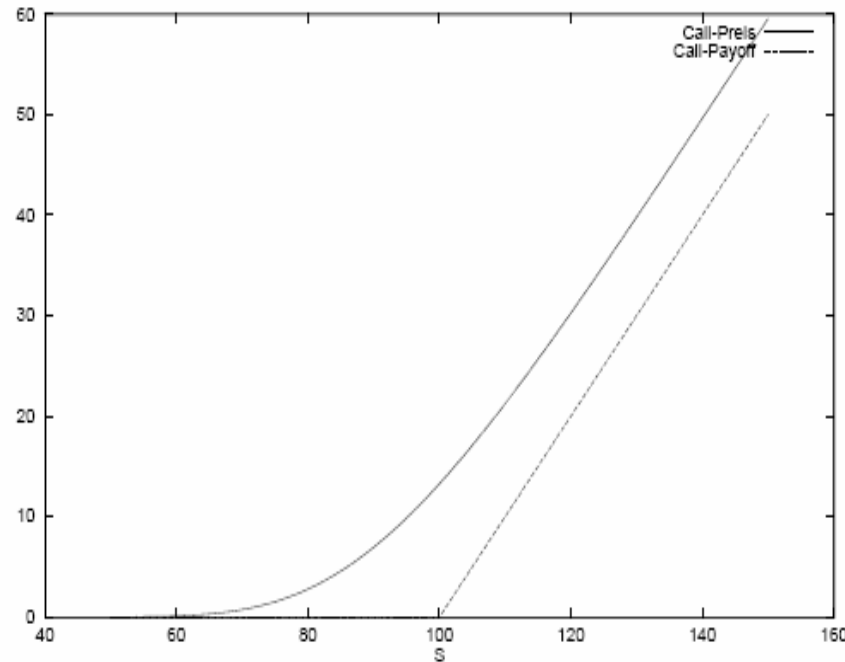


Black-Scholes model: Pricing formula

- Unfortunately **no exact analytic formulae for the value of American calls and puts** have been produced. There are however numerical procedures (e.g. Monte Carlo simulations) that provide solutions.
- Note: Since the **American call price equals the European call price for a non-dividend paying stock**, the BS formula also gives the price of an American call on a non-dividend paying stock. (Remember: Early exercise of an American call on a non-dividend paying stock is never optimal.)

Properties of the Black-Scholes prices

- Black-Scholes price of a call and a put with $E = 100$, $T = 1y$, $r = 10\%$ p.a. and $\sigma = 20\%$ p.a..





Properties of the Black-Scholes prices

- Call
 - When **S becomes very large**, a call option is **almost certain to be exercised**. It then becomes very similar to a **forward purchase** contract with delivery price E.
 - ⌚ Expect the **call price to be** $c_t = S_t - Ee^{-rT}$.
 - This is in fact the call price given by the BS formula since when S becomes very large, both d1 and d2 become very large and since N(x) is the probability that a variable with a standard normal distribution, i.e. N(0,1), will be less than x, N(d1) and N(d2) are both close to one.

- Put
 - When **S becomes very large**, a European put option is **almost certain to be not exercised**.
 - ⌚ Expect the **European put price to be 0**.
 - This is in fact the European put price given by the BS formula since when S becomes very large, both d1 and d2 become very large and since N(x) is the probability that a variable with a standard normal distribution, i.e. N(0,1), will be less than x, N(-d1) and N(-d2) are both close to zero.



Properties of the Black-Scholes prices

- Call
 - When **S becomes very small**, a call option is **almost certain to be not exercised**.
 - ⌚ Expect the **call price to be 0**.
 - This is in fact the call price given by the BS formula since when S becomes very small, both d1 and d2 become very small and since N(x) is the probability that a variable with a standard normal distribution, i.e. N(0,1), will be less than x, N(d1) and N(d2) are both close to zero.

- Put
 - When **S becomes very small**, a European put option is **almost certain to be exercised**. It then becomes very similar to a **forward sale** contract with delivery price E.
 - ⌚ Expect the **European put price to be** $p_t = Ee^{-rT} - S_t$.
 - This is in fact the European put price given by the BS formula since when S becomes very small, both d1 and d2 become very small and since N(x) is the probability that a variable with a standard normal distribution, i.e. N(0,1), will be less than x, N(-d1) and N(-d2) are both close to one.



Properties of the Black-Scholes prices

- Call
 - When σ becomes 0 , the **stock is virtually riskless, its price will grow at rate r to Se^{rT} at time T and the payoff from a call option is $\max(S_T - E, 0) = \max(Se^{rT} - E, 0)$.**
 - **Discounting at rate r** , the value of the call today is $c_t = e^{-rT} \max(Se^{rT} - E, 0) = \max(S - Ee^{-rT}, 0)$.
 - To show that this is consistent with the BS formula, consider first the case where $S > Ee^{-rT}$. This implies $\ln(S/E) + rT > 0$. As σ tends to zero, $d1$ and $d2$ tend to $+\infty$, so that $N(d1)$ and $N(d2)$ tend to 1 and the BS formula becomes $c_t = S_t - Ee^{-rT}$.
 - Next consider the case where $S < Ee^{-rT}$. This implies $\ln(S/E) + rT < 0$. As σ tends to zero, $d1$ and $d2$ tend to $-\infty$, so that $N(d1)$ and $N(d2)$ tend to 0 and the BS formula yields 0.



Properties of the Black-Scholes prices

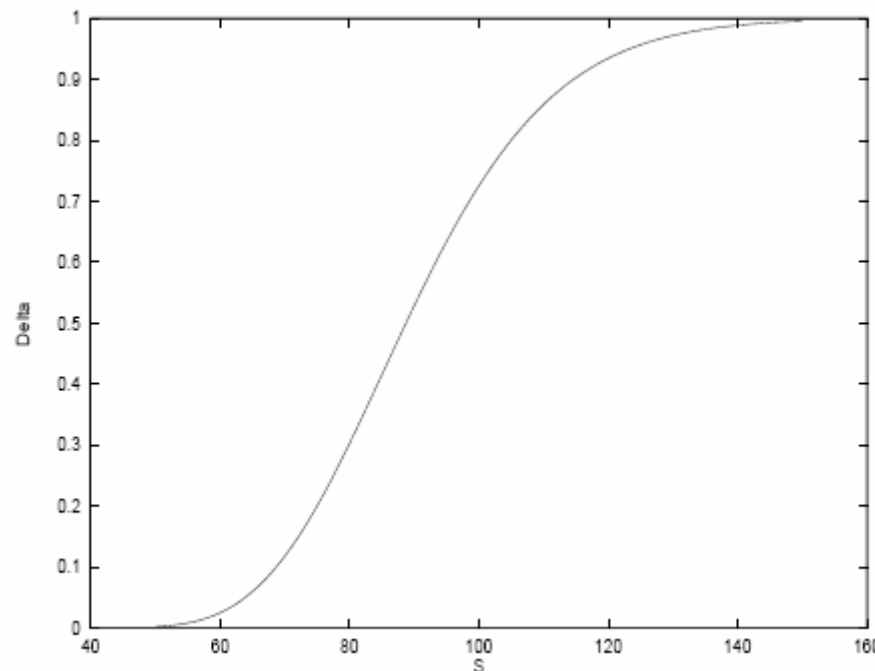
- Put
 - When σ becomes 0 , the **stock is virtually riskless**, its **price will grow at rate r to Se^{rT} at time T** and the **payoff from an European put option is $\max(E - S_T, 0) = \max(E - Se^{rT}, 0)$.**
 - **Discounting at rate r** , the value of the European put today is
$$p_t = e^{-rT} \max(E - Se^{rT}, 0) = \max(Ee^{-rT} - S, 0).$$
 - To show that this is consistent with the BS formula, consider first the case where $Ee^{-rT} > S$. This implies $\ln(S/E) + rT < 0$. As σ tends to zero, d_1 and d_2 tend to $-\infty$, so that $N(-d_1)$ and $N(-d_2)$ tend to 1 and the BS formula becomes $p_t = Ee^{-rT} - S_t$.
 - Next consider the case where $Ee^{-rT} < S$. This implies $\ln(S/E) + rT > 0$. As σ tends to zero, d_1 and d_2 tend to $+\infty$, so that $N(-d_1)$ and $N(-d_2)$ tend to 0 and the BS formula yields 0.

Greeks

- **Delta: change of the option price given a change in the underlying price.**

$$Call : \Delta = \frac{\partial c}{\partial S} = N(d1) > 0 \quad \& \quad Put : \Delta = \frac{\partial p}{\partial S} = N(-d1) < 0$$

- Black-Scholes delta of a call with $E = 100$, $T = 1y$, $r = 10\%$ p.a. and $\sigma = 20\%$ p.a..



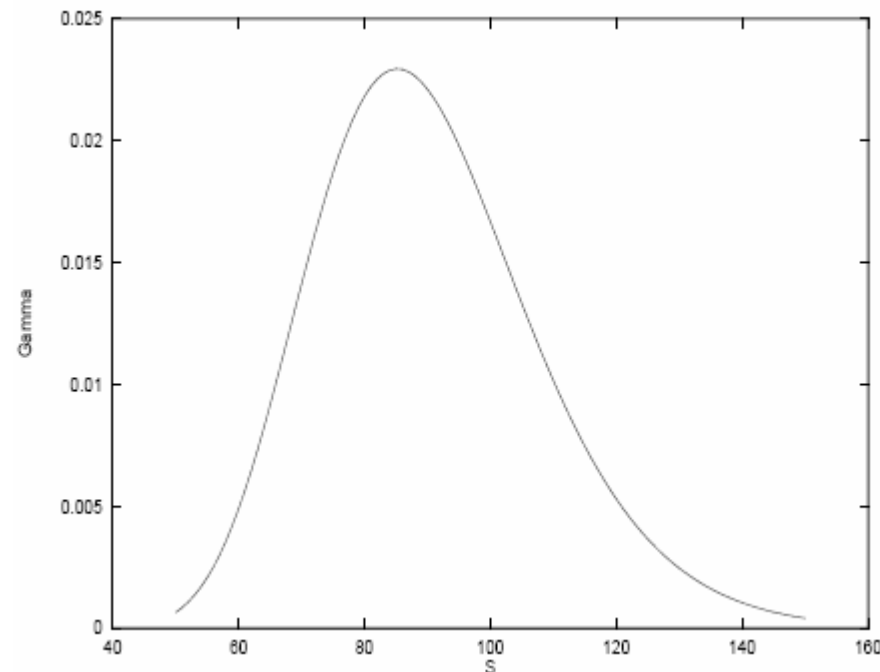
Greeks

- **Gamma: change of the delta given a change in the underlying price.**

$$Call : \Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 c}{\partial S^2} = \frac{N'(d1)}{S\sigma\sqrt{T}} \quad \& \quad Put : \Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 p}{\partial S^2}$$

Note: If gamma is small, delta changes only very slowly.

- Black-Scholes gamma of a call with $E = 100$, $T = 1y$, $r = 10\%$ p.a. and $\sigma = 20\%$ p.a..

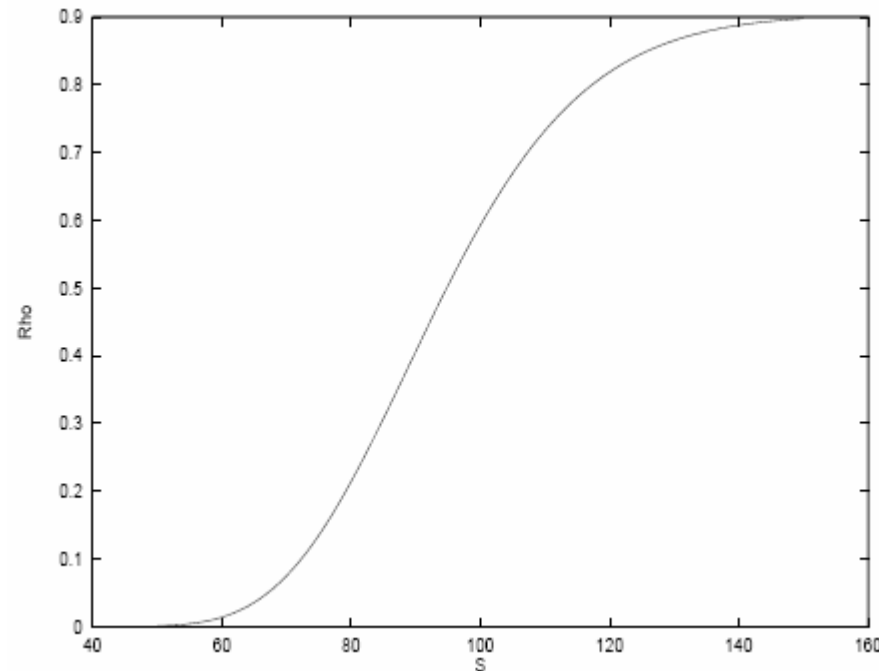


Greeks

- **Rho: change of the option price given a change in the “riskless” interest rate.**

$$\text{Call} : \rho = \frac{\partial c}{\partial r} = ETe^{-rT} N(d_2) \quad \& \quad \text{Put} : \rho = \frac{\partial p}{\partial r}$$

- Black-Scholes rho of a call with $E = 100$, $T = 1y$, $r = 10\%$ p.a. and $\sigma = 20\%$ p.a..



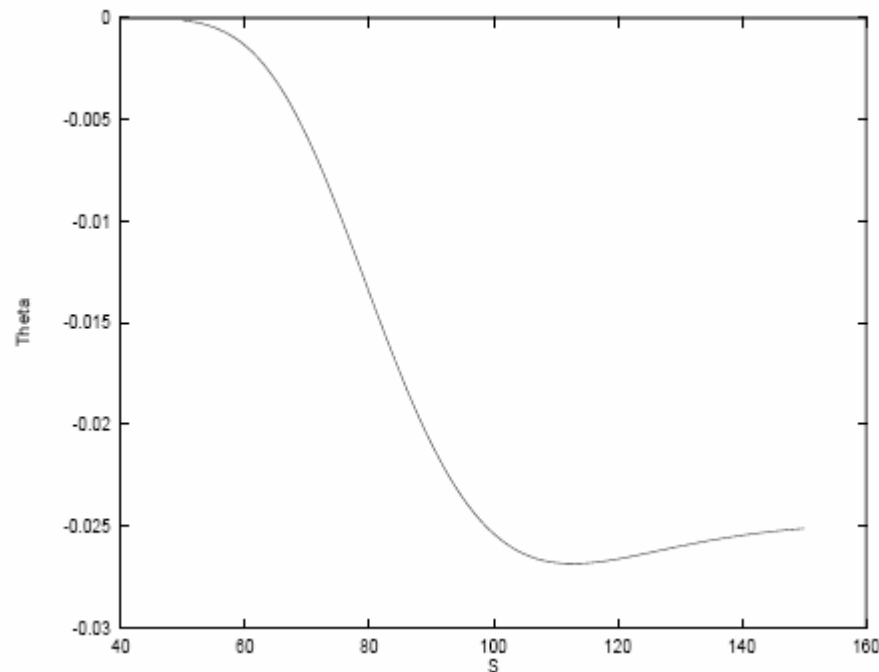
Greeks

- **Theta: change of the option price given a change in t.**

$$Call : \Theta = \frac{\partial c}{\partial t} = -\frac{SN'(d1)\sigma}{2\sqrt{T}} - rEe^{-rT}N(d2) \quad \& \quad Put : \Theta = \frac{\partial p}{\partial t}$$

In other words, theta measures how fast the value of the option changes as time goes by, all other things equal.

- Black-Scholes theta of a call with $E = 100$, $T = 1y$, $r = 10\%$ p.a. and $\sigma = 20\%$ p.a..

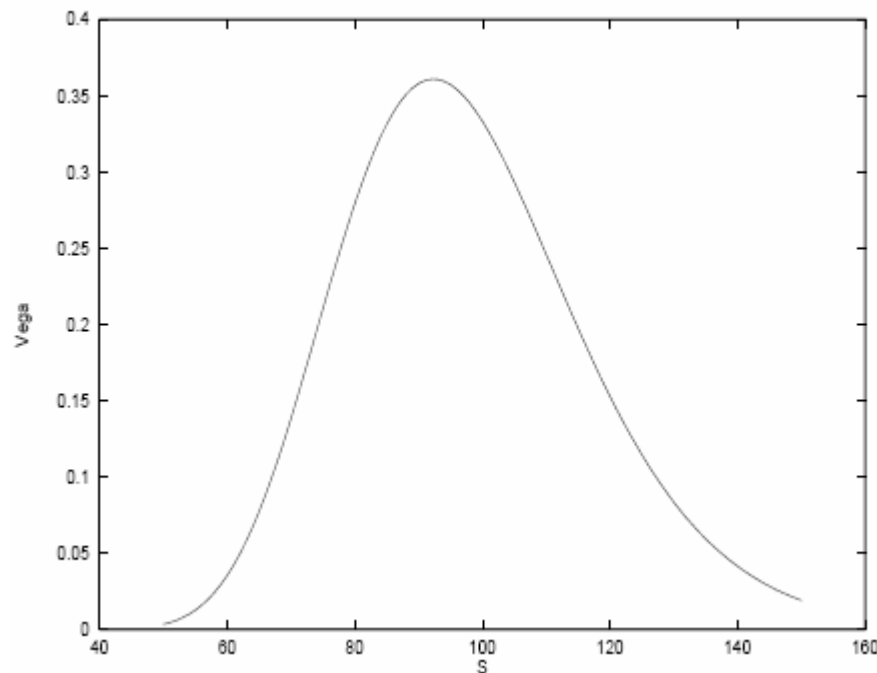


Greeks

- **Vega: change of the option price given a change in the volatility.**

$$\text{Call: } v = \frac{\partial c}{\partial \sigma} = S\sqrt{T}N'(d1) \quad \& \quad \text{Put: } v = \frac{\partial p}{\partial \sigma}$$

- Black-Scholes vega of a call with $E = 100$, $T = 1y$, $r = 10\%$ p.a. and $\sigma = 20\%$ p.a..



Payoffs of path-independent options

- European call and put
- American call and put
- Bermudan call and put ... like American calls and puts but with early exercise allowed only on some pre-specified dates
- Digital / binary options
 - Call $1_{S_T \geq E}$
 - Put $1_{S_T < E}$
 - “Mixed” $1_{a \geq S_T \geq b}$
- Power options
 - Call $\max(0, S_T^n - E^n)$, where $n \in N$
 - Put $\max(0, E^n - S_T^n)$, where $n \in N$

Payoffs of path-dependent options

- Barrier options

- Up and out call

$$\max(0, S_T - E) 1_{\max_{t \in [0, T]} S_t \leq L}, \text{ with } L > S_0$$

- Up and in call

$$\max(0, S_T - E) 1_{\max_{t \in [0, T]} S_t \geq L}, \text{ with } L > S_0$$

- Down and out call

$$\max(0, S_T - E) 1_{\min_{t \in [0, T]} S_t \geq L}, \text{ with } L < S_0$$

- Down and in call

$$\max(0, S_T - E) 1_{\min_{t \in [0, T]} S_t \leq L}, \text{ with } L < S_0$$

Payoffs of path-dependent options

- Lookback options

- Call

$$\max_{t \in [0, T]} S_t - S_T$$

- Put

$$S_T - \min_{t \in [0, T]} S_t$$

- Shout option

$$\max\left(0, \max\left(S_T, S_{t_{shout}}\right) - E\right), \text{ with } t_{shout} \text{ det. by the holder during } T$$

- Average / Asian option

251 $\max\left(0, \left(\prod_{i=1}^n S_{t_i}\right)^{\frac{1}{n}} - E\right)$ or arithm. avg., with all the t_i 's det. at $t = 0$

Payoffs of multi-asset options

- Basket option

$$\max\left(0, \sum_{i=1}^n w_i S_{T,i} - E\right), \text{ where } w_i \text{ are the weights for the assets}$$

- Quanto option

$$X\left(0, S_T^{FC} - E^{FC}\right), \text{ where } X\left[\frac{HC}{FC}\right] \text{ is the exchange rate}$$

- And much, much more.

Payoffs of options on options

- Compound option

$\max(0, c_T - E)$, where c_T is the price of an European call
with maturity $\hat{T} > T$

- Chooser option

$\max(p_T, c_T)$, where p_T and c_T are the prices of an European put
and call with the same underlying, strike price, and
maturity $\hat{T} > T$

- And much, much more.

Overview

- Introduction
- Interest rate swap
- Currency swap
- Pricing swaps with forwards



Definition

- A swap is an agreement to **exchange cash flows at specified future times according to certain specified rules.**



Interest rate swap

- Converting cash flows (investments or liabilities) from
 - fixed rate to floating rate OR
 - floating rate to fixed rateat specified future times.
- Interest rate swaps can be valued as the difference between the value of a fixed-rate bond and the value of a floating-rate bond.

Example

- An agreement by "Company B" to receive 6-month LIBOR and pay a fixed rate of 5% per annum every 6 months for 3 years on a face value (notional principal) of \$100 million.
- Cash Flows to company B

-----Millions of Dollars-----				
Date	LIBOR Rate	FLOATING Cash Flow	FIXED Cash Flow	Net Cash Flow
Mar.1, 1998	4.2%			
Sept. 1, 1998	4.8%	+2.10	-2.50	-0.40
Mar.1, 1999	5.3%	+2.40	-2.50	-0.10
Sept. 1, 1999	5.5%	+2.65	-2.50	+0.15
Mar.1, 2000	5.6%	+2.75	-2.50	+0.25
Sept. 1, 2000	5.9%	+2.80	-2.50	+0.30
Mar.1, 2001	6.4%	+2.95	-2.50	+0.45

Currency swap

- Converting cash flows (investments or liabilities) from one currency in another currency. The coupon payments may be either
 - fixed rate to floating rate,
 - floating rate to fixed rate,
 - fixed rate to fixed rate, OR
 - floating rate to floating rateat specified future times.
- Currency swaps can be valued as the difference between 2 bonds.

Example

- An agreement by "Company B" to pay 11% on a FV of £10,000,000 and receive 8% on a FV of \$15,000,000 every year for 5 years.
- In an interest rate swap the principal is not exchanged. However, in a currency swap the principal is exchanged at the beginning and the end of the swap.

	\$	£
Years	-----millions-----	
0	-15.00	+10.00
1	+1.20	-1.10
2	+1.20	-1.10
3	+1.20	-1.10
4	+1.20	-1.10
5	+16.20	-11.10

Pricing swaps with forwards

- A swap can be regarded as a convenient way of packaging forward contracts.
- The interest rate swap in our example consisted of 6 forward contracts.
- The currency swap in our example consisted of a cash transaction and 5 forward contracts.
- The **value of the swap is the sum of the values of the forward contracts implied by the swap.**
- A swap is worth zero to a company initially. However, at a future time its value is either positive or negative.